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MATHEMATICAL ESSAYS.

By W. LUDLAM,

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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MATHEMATICAL ESSAYS

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PROP. XIII. B. I. OF NEWTON'S PRINCIPIA.

The Essays on Ultimate Ratios and the Power of the Wedge were first published in 1770: that on Newton's second Law of Motion in 1780, the rest are now added.

ESSAY I.

ON ULTIMATE RATIOS.

1. **A** GREAT part of the labour of geometricians has been employed in comparing the areas of curvilinear figures with rectilinear figures, and the areas of curvilinear figures with each other; not merely on account of the importance of the subject itself, but because almost all questions in philosophy can be reduced to a comparison of such areas. For instance; there are no two forces in mechanics however related to each other, but geometricians can feign two figures so circumstanced that their areas shall have the same proportion to each other as the proposed forces; therefore any theorems serving to discover the unknown proportion of those geometrical figures from the assumed circumstances, will discover likewise the proportion of the forces analogous to them. An example of this sort we have in the doctrine of falling bodies, in which the proportion of the right lines described in their descent when the times of their fall are different, is found by comparing the areas of two right angled triangles. For if the perpendicular heights of those triangles be assumed proportional to the respective times of descent and their bases to the velocities acquired in the fall, it is demonstrable that the right lines described in their descent will be proportional

A

portional to the areas of those triangles; and thus the right line described by heavy bodies in their fall, is said to be *represented* by the area of a right angled triangle*.

2. We find Newton frequently prefacing his problems with this expression, *Concessis figurarum curvilinearum quadraturis, requiritur, &c.* that is by feigning some geometrical figure whose area he will show to be analogous to the quantity sought, the problem is reduced to a geometrical one. By this means we are assured that the conditions of the problem involve no inconsistency but that it is possible; and if the art of comparing curvilinear figures be sufficiently understood, the solution itself may be drawn from hence, and perhaps many other circumstances may be inferred, which otherwise would not easily appear.

3. The primary method of comparing the magnitude of rectilinear figures, is by laying them upon one another, it being an axiom that such as agree in all their parts are equal†. Thus all the right lined figures which in the first book of the *Elements* are proved to be equal, are always shown to be the sum or difference of such as would respectively cover each other. This is the direct and best way of reasoning, in which the agreement of the extreme ideas is shown by their continual agreement with intermediate ideas of the same kind: but this method cannot be applied in comparing curvilinear figures with rectilinear ones, because no part of a curve line, can ever coincide with a straight line.

4. To extend geometry to the comparing curvilinear with rectilinear figures the ancients made use of the method of *Exhaustions*‡; an example of which we have in the second proposition of the twelfth book of Euclid. I shall only observe upon it, that the argument made

* Keil's *Physica* Lect. XI. Theorem XVII. *Cotes De descensu gravium*. Mac Laurin's Account, &c. B. II. Chap. 5.

† Axiom 8 and prop. 4. El: I.

‡ See *Archimedes de quadratura parabolæ*, &c.

made use of is what is called *reductio ad absurdum*, which though strictly logical is always tedious, inasmuch as every proposition must be divided into two cases, in one of which you are to show that the former of the quantities to be compared together is not greater than the latter, and in the other that it is not less.

5. To contract this method of reasoning, *Cavalieri* proposed what is called the method of *Indivisibles**, in which he was followed by Dr. *Wallis* and others of the last century. In this method every line was supposed to consist of a number of other lines of the *smallest possible* length; every curve was considered as a polygon each of whose sides is one of those *indivisible* lines; with other like suppositions equally absurd and un-geometrical. These principles soon led their followers into perplexity and oft-times into error, nor was it easy to fix bounds to those liberties when once introduced.

6. To avoid both the tediousness of the antients and the inaccuracy of the moderns, Sir Isaac Newton introduced what he called the method of *prime and ultimate ratios*, the foundation of which is contained in the first Lemma of the first book of the *Principia*. Many difficulties have been started and much controversy raised concerning the proof of it†; all which would have been avoided had either the author or his readers observed that he is in reality laying down the *definition* of a term, (namely, being *Ultimately equal*,) and not *proving a proposition*. Taking therefore this first Lemma for a definition it may be explained, (not *proved*) in the manner following.

7. Let there be two quantities, one fixed and the other varying, so related to each other that *first* the varying quantity continually approaches to the fixed quantity. *Secondly*, that the varying quantity does never reach or pass beyond that which is fixed.

Thirdly,

* See *Geometria indivisibilibus promota*. Ed. 1635.

† See several pamphlets written by Mr. *Robins* and Dr. *Jurin* in the years 1736 and 1740, and other papers in the memoirs of literature for those and the intermediate years.

Thirdly, that the varying quantity approaches nearer to the fixed quantity than by any assigned difference *, then is such a fixed quantity called the *limit* of the varying quantity; or in a looser way of speaking, the varying quantity may be said to be *ultimately* equal to the fixed quantity †. These three properties may be otherwise expressed more distinctly thus. *First*, the difference between the varying quantity and the fixed quantity must continually decrease. *Secondly*, this difference must never become either nothing or negative. *Thirdly*, this difference must become less in respect to the fixed quantity than by any assigned ratio; or the difference between the two quantities must become a less part of the fixed quantity than any fractional part that is assigned, how small so ever the fraction expressing such part may be ‡. Wherever these properties are found, the fixed quantity is called the *limit* of the varying quantity, or the varying quantity is *said* to be *ultimately equal* to the fixed quantity. This last phrase must not be taken in an absolute literal sense, there being no *ultimate state*, no particular magnitude that is the *ultimate magnitude* of such a varying quantity. Under the word quantity in this definition must be included not only numbers, lines, &c. but more especially ratios considered as a peculiar species of quantity; but as the consideration of ratios which have limits is difficult, we shall begin with examples of other quantities.

EXAMPLES

* *Propius accedunt quam pro data quavis differentia.*

† Some writers seem to have dropped the second condition out of their idea of a limit; however Newton plainly supposes it. His words are—*Ultimæ rationes—revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant; et quas propius assequi possunt quam pro data quavis differentia, nunquam vero transgredi,* &c. *Scholium ad Lemma XI. Lib. 1. Princip. pag. 38. Ed. 3.*

‡ See Rudiments of Mathematics, Part II. par. 124.

EXAMPLES IN NUMBERS.

8. Let there be formed a series whose first term is unity, second $\frac{1}{2}$, third $\frac{1}{4}$, fourth $\frac{1}{8}$, and so on, every term being half the preceding term thus; $1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot$

$\frac{1}{8} \cdot \frac{1}{16}$, &c. and let the sum of an indefinite number of terms in this series be considered as continually varying, being continually increased by the accession of a new term; thus the sum of two terms is $1\frac{1}{2}$, of three terms is $1\frac{3}{4}$, of four terms is $1\frac{7}{8}$, &c. I say then that the varying sum of the terms of this series, continually approximates to the fixed number 2 as its limit. For the difference between the sum of one, two, three, four, &c. terms and the number 2 will be

the numbers $1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}$, &c. successively in infinitum. Here it is evident that the terms of this last series which express the successive differences between the growing sum of the terms of the former series and the number 2, *first* continually decrease; *secondly* no term in this series of differences can become either nothing or negative; *thirdly* we may continue this series of successive differences, 'till we come to a term which shall be a less part of the fixed number 2, than any fractional part that can be assigned, or so that this difference shall be less when compared with the number 2, than in any proportion assigned. For instance we may continue this series of differences, 'till we come to a term which shall be less than a $\frac{1}{1000}$ -th

part of the number 2, or less than the $\frac{1}{10,000}$ -th part of the number 2, or 'till you come to a term which shall be less than any other fractional part of the

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number 2 which you please to point out and assign. The number 2 then having the conditions laid down in the definition is to be *called* the limit of the sum of the terms of this infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c.

Any infinite series being proposed, if a number can be pointed out having the conditions before mentioned then such series is said to have a limit.

In the usual phrase of mathematicians, the number 2 in our example would be called the sum of *all* the terms in the proposed series continued *in infinitum*; but in this case the words must not be understood in their literal sense, it being absurd to speak of the sum of *all* the terms of *an infinite* series, that is, of a series which has *no all* to its number of terms. When mathematicians talk of finding the sum of an infinite series, they neither mean the sum of any particular number of terms, nor yet the sum of all the terms, but the *limit* of the sum of the terms as before explained.

9. No number less than 2, for instance $1\frac{1}{2}$ can be taken for the limit; for in this case it will not answer the second condition. In this example the sum of four terms of the series equals $1\frac{1}{2}$, and the sum of five terms exceeds it; therefore the difference between this sum and the number $1\frac{1}{2}$ proposed as a limit is in the former case nothing, and in the latter negative. No number greater than 2, for instance 3, can be taken for the limit, because the last condition will be wanting. For if the sum of any assigned number of terms be always less than 2, that sum must always want more than unity of the number 3, and consequently cannot approach nearer to 3 than by the assigned difference 1.

10. In like manner the sum of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$, &c, continued in infinitum will be $1\frac{1}{2}$; the series of

of successive differences being $\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{18} \cdot \frac{1}{54}$, &c. in infinitum.

11. To give a similar instance in Geometry; Let AI (in fig. 1.) be an indefinite right line, on which set off the equal parts $AB \cdot BC \cdot CD \cdot DE$, &c. on AB , make a square*, on BC a right angled parallelogram whose breadth Bb is half the side AB , on CD a parallelogram whose breadth Cc is half the breadth of the former parallelogram, on DE another whose breadth is half Cc and so on. It is evident that the right lined figure composed of all these parallelograms though infinite in extent from A towards I , is yet finite in area, and is equal to twice the square upon AB ; that is, though there is no limit to the extent of such a figure from A towards I , yet there is a limit to its area in our sense of the word limit; nor is it strange that quantities which have no limits in one respect, should yet have limits in another.

12. To make the example in par. 8. general. Let $a : b$ express the common ratio of any series of numbers in continual proportion whose first term is unity; I say then that if a be greater than b , such a series will have a limit, viz. $\frac{a}{a-b}$, or in other words the sum of all the terms of that series continued in infinitum will be $\frac{a}{a-b}$. For the terms of this series will be $1 \cdot \frac{b}{a} \cdot \frac{b^2}{a^2} \cdot \frac{b^3}{a^3} \cdot \frac{b^4}{a^4}$, &c. whence

The sum of 1 term is 1

2 terms is . . . $\frac{a+b}{a}$

3 terms is $\frac{aa+ab+bb}{aa}$

4 terms is $\frac{a^3+aab+abb+b^3}{a^3}$

Let

* 46 El. I.

Let each of these sums be subtracted from the limit $\frac{a}{a-b}$, and we have the successive differences

$\frac{b}{a-b} \cdot \frac{b}{a-b} \times \frac{b}{a} \cdot \frac{b}{a-b} \times \frac{b^2}{a^2} \cdot \frac{b}{a-b} \times \frac{b^3}{a^3}$, &c. or if n be any assigned number of terms, the difference between the sum of that number of terms and $\frac{a}{a-b}$

will be $\frac{b}{a-b} \times \left(\frac{b}{a}\right)^{n-1}$ *. Whence we may observe, *first* that as the number of terms to be summed up increases, this difference $\frac{b}{a-b} \times \left(\frac{b}{a}\right)^{n-1}$ continually decreases,

* The sum of any particular number of terms n , may be thus found.

Let A, B, C, D, E be any series in continual proportion. Then A is to B as B to C , or as C to D , or as D to E . Therefore A, B, C, D or all the terms except the last, are antecedents; and B, C, D, E , or all the terms except the first, are consequents. Put S for the sum of all the terms, and $S-E$ will be the sum of all the antecedents, and $S-A$ will be the sum of all the consequents. Whence (by 12. El. V.) $A : B :: S-E : S-A$, and $AS-AA=BS-BE$, and $S=\frac{AA-BE}{A-B}$.

Cor. In a decreasing series continued *in infinitum*, the last term E vanishes; and the sum of the whole series is $\frac{AA}{A-B}$.

Put $x=\frac{b}{a}$, and the series before mentioned (viz. $1 \cdot \frac{b}{a} \cdot \frac{b^2}{a^2} \cdot \frac{b^3}{a^3}$, &c.) for n terms will stand thus;

$$1 + x + x^2 + x^3 \dots x^{n-1} = S.$$

Here $A=1$, $B=x$, $E=x^{n-1}$, whence S or the sum of n terms is $\frac{1-x^n}{1-x}$. Now because x is a fraction less than unity, its powers will continually decrease. By increasing n , the number x^n may become less than any assigned fraction; that is when n is infinitely great, x^n vanishes; and we have $S=\frac{1}{1-x}$; and restoring x , $S=\frac{a}{a-b}$, as before.

creases, because $\frac{b}{a}$ being a fraction less than unity, its powers continually decrease. *Secondly*, this difference can never become nothing or negative; the powers of a fraction though they decrease being always real and affirmative. *Thirdly*, this difference may become less in respect of $\frac{a}{a-b}$ than by any assigned proportion.

For $\frac{a}{a-b}$ is to $\frac{b}{a-b} \times \frac{b}{a}^{n-1}$ as a is to $\frac{b}{1} \times \frac{b^{n-1}}{a^{n-1}}$, or as a^n is to b^n or as $\left(\frac{a}{b}\right)^n$ is to 1; but $\frac{a}{b}$ is greater than unity,

therefore the powers of $\frac{a}{b}$ continually increase, and by increasing the index n the number $\left(\frac{a}{b}\right)^n$ may become

greater than any assigned number, consequently $\left(\frac{a}{b}\right)^n$ may become greater in respect of 1, or 1 less in respect of $\left(\frac{a}{b}\right)^n$ than by any assigned ratio; therefore the

difference $\frac{b}{a-b} \times \frac{b}{a}^{n-1}$ may become a less part of $\frac{a}{a-b}$

than any assigned fractional part. The quantity $\frac{a}{a-b}$

therefore having these three properties agrees with the definition of a limit of the growing sum of the terms of the series, or we may say (in the language of mathematicians) that it is equal to the sum of all the terms continued in infinitum.

13. What has been thus proved may be shown more concisely by dividing a by $a-b$ in the manner of compound division in arithmetic; for the quotient will be the very series proposed in the example, and it may be further confirmed by multiplying that series

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series by $a-b$, for the product will be a , all the terms except the first destroying one another; therefore a divided by $a-b$ is equal to that whole series *.

14. We may observe that if every term of the foregoing series be multiplied by any number c it will become $c + \frac{bc}{a} + \frac{b^2}{a^2} \times c + \frac{b^3}{a^3} \times c$, &c. that is, a series of numbers in continual proportion whose common ratio is that of a to b as before, but whose first term is c . It is evident therefore, that in this case, to find the sum of all the terms we must multiply the former limit by c , that is, it will be $\frac{ac}{a-b}$.

15. *Example 1.* Let the first term be 1, and the common ratio 10 to 1, then the several terms of the series are $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, &c, or if we express them decimally

the 1st term is 1,

2 ^d	0,1
3 rd	0,01
4 th	0,001

their sum $\overline{1,111}$, &c. $= \frac{10}{10-1} = \frac{10}{9}$.

Example 2. Let the first term be 1, the common ratio 9 to 1, and the terms expressed decimally are

the

* Hence again we may find the sum of n terms in this series. For putting $x = \frac{b}{a}$ as before, the last or n th term will be x^{n-1} , and the remaining terms of the series will be $x^n + x^{n+1} + x^{n+2} + x^{n+3}$, &c. $= x^n \times \overline{1+x+x^2+x^3+x^4}$, &c. $= x^n \times \frac{1}{1-x} = \frac{x^n}{1-x}$; Subtract this from $\frac{1}{1-x}$ or the sum of all the terms, and there will remain the sum of the first n terms $= \frac{1-x^n}{1-x}$ as before.

the 1st term	1,
2 ^d	0,111111
3 ^d	0,012345
4 th	0,001371
5 th	0,000152
6 th	0,000017

$$\text{their sum } 1,124996 = \frac{9}{8} = 1,125.$$

16. Before we proceed to consider ratios that have limits we must learn to distinguish between the terms of a ratio and the ratio itself. The terms of a ratio may vary through all degrees of magnitude, and yet the ratio itself continue invariable. Let AQ (fig. 2.) be an indefinite right line, AI another intersecting it in a given angle at A , and on any point P in AQ draw PM at right angles to AQ meeting AI in M . Let the point P move upon the right line AQ so that AP and PM may increase and decrease. The lines AP and PM in this case may vary through all degrees of magnitude, may become greater or less than any assigned line; that is, in the common language of mathematicians, may become infinitely great or infinitely little, yet the ratio of AP to PM is invariably the same, because the triangles APM are always similar.

In this instance then, varying the terms does not vary the ratio at all; in other instances varying the terms in infinitum does likewise vary the ratio in infinitum; and there are instances where although changing the terms does change the ratio, yet the ratio never varies beyond certain limits, even though the terms themselves should increase or decrease in infinitum. The following example of this is borrowed from *Saunderson's Algebra*, art. 336.

17. Let x be any varying quantity; make $4xx + 3x = A$ and $2xx + x = B$, then will A and B also be varying quantities as depending upon x ; when x vanishes A and B will both vanish, and when x is infinite

nite they will both be infinite. I say that the ratio of A to B while x decreases in infinitum approximates to the ratio of 3 to 1 as its limit. For *first* A is to B as $4xx+3x$ is to $2xx+x$ or as $4x+3$ is to $2x+1$. Here it is easy to see that as x decreases the quantities $4x$ and $2x$ decrease, and consequently the ratio of $4x+3$ to $2x+1$ or A to B approaches to that of 3 to 1. *Secondly*, the ratio of A to B can never reach or exceed that of 3 to 1. For $6xx+3x$ is to $2xx+x$ as 3 is to 1, but $4xx+3x$ is a less quantity than $6xx+3x$, therefore $4xx+3x$ is to $2xx+x$ or A is to B always in a less ratio than that of 3 to 1. *Lastly*, the ratio of A to B will approach nearer to that of 3 to 1 than by any assigned difference. For in the terms of this ratio $4x+3$ and $2x+1$, the varying parts $4x$ and $2x$ by diminishing x may become less than any assigned quantity while the other parts 3 and 1 remain the same; therefore the ratio of A to B will approach nearer to the ratio of 3 to 1, than by any assigned difference.

In like manner the ratio of A to B , while x increases in infinitum approximates to the ratio of 2 to 1 as its limit. For since A is to B as $4x+3$ is to $2x+1$ or as $4+\frac{3}{x}$ is to $2+\frac{1}{x}$ it is easy to see that as x increases the quantities $\frac{3}{x}$ and $\frac{1}{x}$ decrease and consequently the ratio of $4+\frac{3}{x}$ to $2+\frac{1}{x}$ or A to B approaches to the ratio of 4 to 2 or 2 to 1. Again; the ratio of A to B can never be less (that is nearer to the ratio of equality) than the ratio of 2 to 1. For $4xx+2x$ is to $2xx+x$ as 2 is to 1, but $4xx+3x$ is a greater quantity than $4xx+2x$; therefore $4xx+3x$ is to $2xx+x$ or A is to B always in a greater ratio than that of 2 to 1. *Lastly*; the ratio of A to B will approach nearer to that of 2 to 1, than by any assigned

signed difference. For in the terms of this ratio $4 + \frac{3}{x}$ and $2 + \frac{1}{x}$, the variable parts $\frac{3}{x}$ and $\frac{1}{x}$, by increasing x may become less than any assigned fraction while the parts 4 and 2 remain the same; therefore the ratio of A to B will approach nearer to the ratio of 4 to 2 or 2 to 1 than by any assigned difference.

Here then we see that though diminishing x and consequently diminishing the terms A and B increases their ratio, and contrariwise increasing these terms by increasing the quantity x decreases their ratio, yet there is a limit both to the increase and decrease of this ratio, although there is none to the terms themselves that compose it.

18. The ratio of 3 to 1 which limits the ratio of A to B when these terms decrease in infinitum is called the *ultimate ratio* of the evanescent quantities A and B ; and it is likewise called, but more rarely, the *first ratio* of the nascent quantities A and B . The ratio of 2 to 1, which is their other limit, is called the *ultimate ratio* of the quantities A and B increasing in infinitum.

18. Another example of a similar nature we have as follows. Let x be a varying quantity, and d a constant one, then will $x+d$ and x be two varying quantities capable of all degrees of magnitude. I say that the ratio of $x+d$ to x , while x increases will continually decrease, but not beyond a certain limit, which limit is the ratio of equality. On the contrary, if x decrease, the ratio of $x+d$ to x will continually increase more and more in infinitum and never come to a limit. For $x+d$ is to x as $1 + \frac{d}{x}$ is to 1; now as

x increases, the fraction $\frac{d}{x}$ decreases and may become less than any assigned fraction; but the number 1 which is the other part of the antecedent of this ratio remains

remains the same, so likewise does the consequent of this ratio, therefore the ratio of $1 + \frac{d}{x}$ to 1 continually approximates to the ratio of equality. *Secondly*; the fraction $\frac{d}{x}$ has always some magnitude, therefore the antecedent $1 + \frac{d}{x}$ is ever greater than the consequent 1; therefore the ratio of $1 + \frac{d}{x}$ to 1 can never reach the ratio of equality, much less pass it so as to become a ratio *minoris inæqualitatis*, or a ratio in which the antecedent is less than the consequent. *Lastly*; the varying fraction $\frac{d}{x}$, as x increases, will become less than any assigned fraction, while the other part of the antecedent and likewise the consequent of this ratio remains the same. Therefore this ratio of $1 + \frac{d}{x}$ to 1 will approach nearer to the ratio of equality than by any ratio that can be assigned, how small soever such ratio may be; therefore the ratio of equality is the limit of the ratio of $1 + \frac{d}{x}$ to 1, and consequently the limit of the ratio of $x + d$ to x which continually decreases while the terms which compose this ratio continually increase in infinitum.

If x decreases then $\frac{d}{x}$ will increase and may become greater than any assigned number, how great soever that number may be, while the other part of the antecedent and likewise the consequent remain fixed, each being unity; therefore the ratio of $1 + \frac{d}{x}$ to 1, and consequently the ratio of $x + d$ to x , as x decreases, will become greater than any assigned ratio whatever,
and

and so has no limit to its increase. Here we have an instance of a ratio that has a limit to its decrease, but none to its increase.

We may observe that though the terms of this ratio, viz. $x+d$ and x , never get nearer to each other, their constant difference being d , yet the ratio of the terms approximates to the ratio of equality; though the terms get no nearer in their difference, yet (if the expression may be allowed) they get nearer in their ratio. The *difference* of the terms and the *ratio* of the terms, are ideas very distinct from each other, and in no wise so connected, but that one may vary while the other is constant. In this instance the ratio varies, but the difference of the terms is constant; on the contrary, the difference between the two lines AP and PM in fig. 2. (see par. 16.) continually varies, but the ratio of those lines is invariable.

19. Although the ratio of equality may be strictly called the *limit of the varying ratio* of the quantities $x+d$ and x , yet the terms of this ratio can never be strictly said to be equal, no not *ultimately equal*, since that plainly supposes an *ultimate state* in which they are equal; no nor equal when they *vanish into infinity*, or when they step out of finite existence into infinity. There is no finite quantity next to infinity; no number (for instance) which is the next number to infinity, and therefore no step out of finite into infinite*. Nor can we say in strictness that two infinitely great numbers with a finite difference are equal, it being a proposition plainly absurd and contradictory. There is no such thing in nature as any infinitely great number; and it is contradictory to say of any two numbers, both that they have a difference, and that they are equal. Whoever considers that the idea of infinity is a general or abstract idea, that the idea
of

* Neither is there any step out of a *state of nothingness* into finite existence. There is no fraction so small as to be the *very next* fraction to nothing. No fraction can ever be assigned so small, but another fraction may be assigned that is smaller.

of number is always particular, that infinity is a *property* of number*, a *property* of extension, &c. not any number, not any extension, &c. itself, will easily see that these and such-like expressions can have no *literal* meaning†. As for the metaphorical use of them to avoid circumlocution or the introduction of new terms, it may be allowed (when once the literal meaning has been explained) on this as well as numberless other occasions, both in science and common life.

20. When the difference between any two quantities decreases so as to become a less fractional part of one of them than any assigned fractional part whatever; or when the difference between the terms of a ratio becomes less in respect of one of them (the greater

* See Locke. B. II. Ch. 16 at the end. Also Ch. 17.

† By saying that number or that extension is infinite, we mean only to assert the impossibility of limiting the increase of number or the increase of extension: we mean to assert the absurdity of fixing upon any particular number how great soever, or any particular extension how large soever, as the biggest possible number or the greatest possible extension. There is no such thing existing as a number actually infinite, or an infinite right line. Euclid requires you to allow the possibility of producing a right line as far as ever he is pleased to direct, or as some would say, the possibility of producing it *in infinitum*, but he makes no propositions about infinite right lines — *ἐκβαλλόμεναι αἱ δύο αὐταὶ εὐθεῖαι ἐν ἄπειρον* — in the twelfth axiom which is translated by Commandine, *rectæ lineæ illæ in infinitum producæ*, is rendered by Dr. Simson — these straight lines being continually produced: So likewise prop. 12. El. I. *Ἐπὶ τῇ δοθεῖσῃ εὐθεῖᾳ ἄπειρον*, — translated by Commandine — *super data recta linea infinita* — is rendered by Dr. Simson — upon a straight line of an unlimited length. The term *infinite* has been so abused that it can hardly be admitted in mathematical writings any longer; and it is high time to drop it, when authors talk of adding, subtracting, &c. *infinites and infinitesimals* as familiarly as if they were common numbers.

It may be proper to observe here, that as number and extension have no limit to their increase, so neither have they any limit to their decrease. It is absurd to fix upon any particular fraction as the least possible number, or to fix on any particular line as the least possible extension — There is *no least possible* — There exists no such thing as a fraction infinitely small, or a line infinitely little; nor did Euclid or Archimedes or any of the antient geometers ever suppose it.

greater for instance) than by any assigned ratio (according to the third condition required in par. 7.) we shall express this for the future by saying that such difference *vanishes* in respect of that (*greater*) quantity.

21. Now if the difference between two terms vanishes in respect of one of them, it will also vanish in respect of the other. It is true, that at any assigned time, when the terms have a particular magnitude, the difference between the terms will always be a less fractional part of the greater term than it is of the less term; but as this difference continually decreases it will become the same fractional part of the less term that it was before of the greater term. For instance: let the lesser of the proposed terms be fixed, but let the greater be variable; let the less term be always 99, but the greater term at a certain point of time be 100; their difference at that particular time is 1, which is a $\frac{1}{99}$ th part of the less term, but is only

an $\frac{1}{100}$ th part of the greater term. Now as the difference between the terms is supposed to continually decrease (so as to become equal to, or less than any proposed fractional part of the greater term) let the difference sink to 0,99 (so as to become an $\frac{1}{101}$ th part

of the greater term) and it will now be just an $\frac{1}{100}$ th part of the less term; that is, the difference will now be the same fractional part of the less term, that it was before of the greater term. The terms which before were 100 and 99, will now be 99,99 and 99, and their difference which before was 1, will now be 0,99, or an $\frac{1}{100}$ th part of 99: thus then if the dif-

ference be at any time an $\frac{1}{100}$ th part of the greater
B
term,

term, it will (afterwards) also be an $\frac{1}{100}$ th part of the less term; if the difference become a $\frac{1}{1000}$ th part of the greater term, it will also become a $\frac{1}{1000}$ th part of the less term; and if the difference between the two quantities vanishes in respect of one of them, it does also vanish in respect of the other.

22. We may hence enlarge our definition, par. 7. and in laying down the third condition, say, the difference must vanish in respect *either* of the fixed, or of the varying quantity, since one of these implies the other.

23. What was laid down in par. 21. is true, though neither of the terms be fixed, but both of them increase or decrease without limit perpetually. In the instance before considered, par. 18. in which the terms were $x+d$ and x , and where x continually increasing, the terms themselves do both of them continually increase. In this instance if the difference vanishes in respect of one of the terms it will also vanish in respect of the other. Let x be to d at any assigned point of time as n is to 1; and d at that instant of time will be the $\frac{1}{n+1}$ th part of the greater term,

and the $\frac{1}{n}$ th part of the less term; now though the cotemporary value of these fractions can never be equal, yet in succession (as x and consequently n increases for ever without limit) the value of the latter fraction will become whatever the former has been.

If $\frac{1}{n+1}$ becomes less than any assigned fraction so will

$\frac{1}{n}$ likewise; and thus if the difference d vanishes in respect of one of the terms $x+d$ (because x increases without

without limit) so will d vanish also in respect of the other term x . The same would be true if the terms were x and $x-d$, all things being as before.

24. Again; if x by decreasing vanishes in respect of some fixed quantity a , then will x multiplied by any given number n or nx vanish in respect of a ; or which is the same, if x vanish in respect of a , so likewise will the quantity nx , bearing to x the assigned (or as it is called the finite) ratio of n to 1. For though at any particular point of time, x is a smaller fractional part of a than nx , yet x can be no assigned fractional part of a whatever, but by further diminishing x , the quantity nx may become the same fractional part of a that x was before. If then x may become equal to, or less than any assigned fractional part of a , so likewise may nx ; that is, if x vanishes in respect of a , so likewise does nx .

25. And for a like reason, if x vanish in respect of any quantity a , it will likewise vanish in respect of that quantity multiplied or divided by any number; or it will vanish in respect of a quantity, bearing to a any assigned ratio, as that of m to 1. Thus, if x vanish in respect of a , it will also vanish in respect of $3a$, $2a$, or $\frac{1}{2}a$, or $\frac{1}{3}a$. If any line vanish in respect to the diameter of a circle, it will likewise vanish in respect of the radius. For whatever part of the diameter, the line x may be at any assigned point of time, let x further decrease till it be half what it was at that assigned time, and it will now be the same part of the radius that it was before of the diameter. By these rules we may frequently find when one quantity vanishes in respect of another proposed quantity.

26. We have seen before, that the ratio which two quantities bear to each other, may have a limit, although the terms themselves may increase or decrease perpetually without limit. If the terms approximate to each other, if the less never pass the greater, and lastly, if their difference vanish in respect of either term*, then the limit of their varying ratio is that of equality;

* See par. 20. 21. 23.

equality; and this, whether the terms themselves are such as by increasing do both of them become greater than any assigned quantity, or which is more common, such as by decreasing become less than any assigned quantity, or as it is usually called, *infinitely little*. Because the idea of the terms of a ratio is less abstract, than that of the ratio itself, it is more usual to say that the terms themselves in this case are ultimately equal, though strictly it is the ratio of the terms only that comes to a limit; for the terms themselves are here supposed to increase or decrease without limit, or as we commonly say, in infinitum.

27. That the difference between two quantities may thus vanish in respect of those quantities; even though the quantities themselves do vanish in respect of some assigned (or finite) quantity, we shall show by the following examples.

LEMMA.

28. The angle of reflection is equal to the angle of incidence*.

PROBLEM.

29. Let a ray of light parallel to the axis EC (fig. 3.) of a spherical concave speculum, fall on a given point A : to determine the intersection of the reflected ray with the axis of the speculum.

Let AC be the speculum, E the center, EC the axis, QA the incident ray parallel to the axis EC , and meeting the speculum in A , let AT be the reflected ray intersecting the axis in T ; draw the radius EA to the point of incidence, and because the angles AET and QAE †, also QAE and EAT ‡ are equal, the triangle EAT is isosceles. From T draw TD , perpendicular to EA , and it will bisect the radius EA in D ||. Lastly draw AP the sine of the arch AC whose versed sine is PC . Then we have $EP : EA :: ED : ET$; call $EA = r$, $PC = v$, and $ED = \frac{1}{2}r$, therefore $r - v : r :: \frac{1}{2}r$

* Smith's Optics. Art. 8.

† 29. El. I.

‡ Lemma, par. 28.

|| 26. El. I.

$\therefore \frac{1}{2}r : \frac{\frac{1}{2}rr}{r-v} = ET$, and v vanishing $ET = \frac{1}{2}r$. There-

fore bisect the radius EC in F , and if the incident ray QA be supposed always parallel to the axis EC , and continually to approach to it, the intersection T of the reflected ray, with the axis, will continually approach to the point F , as its limit.

30. This is sometimes proved thus, $EP : EA :: ED : ET$, but EP is to EA , ultimately in a ratio of equality*, therefore ultimately $ET = ED$, or half the radius. This will in some sort appear from the nature of the figure itself; for while the incident ray approaches to the axis, the base EA of the isosceles triangle EAT continues the same, but its height TD continually diminishes; therefore the sides ET and TA will at last coincide with the base EA , and because they are equal, ET will at last be equal to half the base, or half the radius†.

31. But as this way of reasoning may not seem very strict, to show the case clearly, we will prove, that the fixed quantity $EF = \frac{1}{2}r$, is the limit of the varying quantity $ET = \frac{\frac{1}{2}rr}{r-v}$.

The quantity ET always exceeds the limit EF , their difference FT being $= \frac{\frac{1}{2}rr}{r-v} - \frac{1}{2}r = \frac{\frac{1}{2}rv}{r-v}$. Now, *first*; this difference continually decreases as (the arch AC , and consequently) v decreases: *secondly*; it never becomes negative: *thirdly*; it vanishes in respect of the varying quantity ET . For $ET : FT :: \frac{\frac{1}{2}rr}{r-v} : \frac{\frac{1}{2}rv}{r-v} :: r : v$, therefore as v vanishes in respect of r , so does FT in respect of ET , the varying quantity; and therefore (as was before shown in par. 21.) FT vanishes in respect of the fixed quantity EF .

32. We see then, that although the point T , or intersection

* See par. 39. † Par. 42.

tersection of the reflected ray with the axis, continually recedes from the speculum, while the incident ray by moving parallel to itself, approaches to the axis, yet this point cannot recede in infinitum, not even though the incident ray be supposed to approach infinitely near the axis. The point F is the *limit* to which the point of intersection may *approach* nearer than by any assigned distance, *beyond* which it cannot *pass*, and *at* which it can never *arrive*, 'till (as we may say) the very intersection itself has lost its being, by the coincidence both of the incident and reflected ray with the axis.

33. This point F is called the *focus* of parallel rays, and the interval FT the *aberration* of the reflected rays from their *geometrical focus*. Now, although FT vanishes in respect of EF , or half the radius, yet it does not vanish in respect of PC the versed sine. On the contrary, it approximates to half the versed sine as its limit, or to speak more strictly, it is the ratio of FT to $\frac{1}{2}PC$, that approximates to the ratio of equality. For the aberration FT being equal to $\frac{\frac{1}{2}rv}{r-v}$,

and $\frac{1}{2}PC$ being equal to $\frac{1}{2}v$, their difference is $\frac{\frac{1}{2}vv}{r-v}$,

which never becomes negative; moreover, $\frac{\frac{1}{2}rv}{r-v} : \frac{\frac{1}{2}vv}{r-v}$

:: $r : v$. Therefore as v vanishes in respect of r , so does the difference between the aberration and half the versed sine, in respect of the aberration, and consequently in respect of half the versed sine too. Nor is it strange that the aberration which vanishes in respect of the fixed quantity $\frac{1}{2}r$, should yet come to a limit with the varying quantity $\frac{1}{2}v$, seeing this last quantity does likewise vanish in respect of $\frac{1}{2}r$.

34. If we were to represent this in the common concise way of speaking, we should say, that the aberration $\frac{\frac{1}{2}rv}{r-v}$, when v vanishes, becomes $\frac{\frac{1}{2}rv}{r}$, or $\frac{1}{2}v$,

seem-

seeming to strike v out of the denominator of the fraction, as being nothing, and yet retaining it in the numerator as being something. Indeed, if v in the numerator be considered as nothing, the whole expression vanishes, and the aberration is nothing, or rather there is no aberration; and thus the whole design of finding out this aberration falls to the ground.

It is from conciseness of expression in such cases as this, that mathematicians have been accused of unfair proceedings; of making the same quantity in the same expression, *to vanish or not to vanish*, as suits the conclusion intended to be drawn from it; of considering quantities as something in the beginning of a demonstration, rejecting such of them as they please at the end of it, and yet retaining the conclusion drawn from the original supposition*.

35. Another proof of the concise kind is as follows. $EA:EP::ET:ED$, whence by division of proportion $EA:(EA-EP)PC::ET:ET-ED=ET-EF=FT$; alternately $EA:ET::PC:FT$, but EA is to ET ultimately as 2 is to 1†, therefore ultimately $2:1::PC:FT$, therefore $FT=\frac{1}{2}PC$.

36. But to proceed. The intire aberration is as we have seen $\frac{\frac{1}{2}rv}{r-v}$; from this subtract $\frac{1}{2}v$, and the

difference $\frac{\frac{1}{2}vv}{r-v}$ may be called the *error of geometrical*

aberration. Now this quantity, though it vanish in respect of the whole aberration, yet comes to a limit

with respect to the quantity $\frac{vv}{2r}$, or a third propor-

tional to the diameter and versed sine, or to use a more strict way of speaking, this error $\frac{\frac{1}{2}vv}{r-v}$ is to

$\frac{vv}{2r}$ ultimately in a ratio of equality: for the whole

error

* See Analyst, Sect. XIV, and XV. † See par. 42.

error of the aberration being $\frac{\frac{1}{2}vv}{r-v}$, and the third proportional aforesaid being $\frac{vv}{2r} = \frac{\frac{1}{2}vv}{r}$, their difference is $\frac{\frac{1}{2}v^3}{r \times r - v}$, and the whole error is to this difference as r is to v , therefore this difference vanishes in respect of the whole error, and consequently in respect of $\frac{\frac{1}{2}vv}{r}$. Thus then this third proportional to the diameter, and versed sine, may be called *the geometrical error of the geometrical aberration*. In like manner we may determine the error of this error, and so on*.

37. To give an example of all this in numbers, let the radius of the speculum be 30 inches, the arch AC eleven degrees and twenty nine minutes, whose versed sine is 2, when the radius is 100, consequently when the radius is 30 inches, this versed sine is 0,6 of an inch. We have therefore $r=30$, $v=0,6$, $\frac{1}{2}rr=450$, $r-v=29,4$, $\frac{\frac{1}{2}rr}{r-v} = \frac{450}{29,4}$, which (by common division) is 15,3061224 = ET , the whole distance of the point of intersection from the center, and this exceeds $EF = \frac{1}{2}r = 15$, or the distance of the geometrical focus by 0,3061224. Again, this quantity which is the whole aberration exceeds the geometrical aberration or $\frac{1}{2}v = 0,3$ by 0,0061224. Again, the diameter being 60, and the versed sine 0,6, we have $60 : 0,6 :: 0,6 : 0,006 =$ the third proportional to the diameter and versed sine; but 0,0061224, which is the whole error of the geometrical aberration, exceeds this third proportional by 0,000124, and so on.

38. All this will be confirmed by dividing the numerator of the original fraction $\left(\frac{\frac{1}{2}rr}{r-v}\right)$ by its denominator in the manner of compound division in arithmetic, and so throwing the fraction into an infinite series,

* *Neque novit natura limitem.* Newton.

series. For the quotient will be $\frac{1}{2}r + \frac{1}{2}v + \frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$, &c. where the law of the series is manifest, each term being to the subsequent one always in the ratio of r to v .

Let then v vanish in respect of r , and all the terms of the series will vanish in respect of the first term; that is, the sum of all the terms will in this case approximate to $\frac{1}{2}r$ as the limit; therefore the geometrical focus is $\frac{1}{2}r$.

Subtract now $\frac{1}{2}r$ from the whole series, expressing the distance ET , and we have the whole aberration $FT = \frac{1}{2}v + \frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$, &c. Let now v again be supposed to vanish in respect of r , and all the terms in this series vanishing in respect of the first, we have $\frac{1}{2}v$ for the geometrical aberration.

Once more; subtract $\frac{1}{2}v$ from the series expressing the whole aberration, and we have $\frac{\frac{1}{2}v^2}{r} + \frac{\frac{1}{2}v^3}{r^2} + \frac{\frac{1}{2}v^4}{r^3} + \frac{\frac{1}{2}v^5}{r^4}$ for the whole error of the aberration. Let v again be supposed to vanish, and we have $\frac{\frac{1}{2}v^2}{r}$ for the limit of the error of the aberration, and so on.

Assuming $\frac{1}{2}r = 15$, and the ratio of r to v that of 100 to 2, as before

we have $\frac{1}{2}r = 15$, *Geometr. Focus.*

100 : 2 :: 15,0 : to $\frac{1}{2}v = 0,3$ *Geometr. Aberr.*

100 : 2 :: 0,3 : to $\frac{\frac{1}{2}v^2}{r} = 0,006$ *Geometr. Error.*

100 : 2 :: 0,006 : to $\frac{\frac{1}{2}v^3}{r^2} = 0,00012$

100 : 2 :: 0,00012 : to $\frac{\frac{1}{2}v^4}{r^3} = 0,0000024$.

Sum of the terms = $ET = 15,3061224$,

which

which is true to seven places of decimals, as was before found by division.

PROPOSITION.

39. In a circle whose center is C , (fig. 4.) radius CA , diameter Aa , let AB be the chord, FB the sine, AD the tangent of the arch AB . I say, that while the arch AB continually decreases without limit; *first*, the sine continually approximates to the tangent; *secondly*, the sine never exceeds the tangent; *thirdly*, their difference will vanish in respect of either sine or tangent.

First, $BF : DA :: CF : CA$; but while the arch AB decreases, CF approximates to CA ; therefore BF approximates to DA . *Secondly*, CF can never exceed CA , therefore BF can never exceed DA . *Thirdly*, the arch continually decreasing without limit, vanishes (in our sense of the word, par. 20.) in respect of the diameter which is fixed; therefore the chord AB , which is less than the arch, likewise vanishes in respect of the diameter Aa ; but $Aa : AB :: AB : AF$; therefore if AB vanishes in respect of Aa , AF will vanish in respect AB , and much more will AF vanish in respect of Aa : but if AF vanish in respect of Aa , it will also vanish in respect of $\frac{1}{2}Aa$ or CA (by par. 25.). Now $CA : CF :: AD : FB$, and by division of proportion, $CA : AF :: AD : AD - FB$. As therefore AF vanishes in respect of CA , so does $AD - FB$ (the difference of the sine and tangent) vanish in respect of AD the tangent, and consequently in respect of FB the sine, (by par. 23.).

We see then, that while the arch continually decreases without limit, the sine approximates to the tangent; *secondly*, the sine never exceeds the tangent; *thirdly*, although the sine and tangent both vanish in respect of the radius, yet their difference vanishes in respect of these quantities themselves. Therefore the ratio of equality is the limit of the varying ratio,

which

which the line and tangent have to each other, while they both decrease perpetually without limit; or it is their *ultimate ratio*; or we may say that they are *ultimately equal*.

40. The sine is less than the chord; for in the right angled triangle AFB , the side BF is less than the hypotenuse AB : the chord is less than the arch; this is self-evident, the chord being a straight line, and the arch a curve, both terminated by the same points A and B . The arch is less than the tangent: for from the point D draw another tangent to the circle in H ; and the lines ABH , ADH will be terminated by the same points A and H , and will have the concavities turned the same way; therefore the included arch ABH will be less than the sum of the two equal tangents DA and DH^* ; consequently, half that arch, or AB , will be less than half the sum of the tangents, or AD .

41. *Cor. 1.* Hence, the sine, chord, arch, and tangent, are all ultimately in a ratio of equality. This may appear, because the chord and arch are included between the sine and tangent, but perhaps more plainly thus. Of these four quantities, *viz.* the sine, chord, arch, and tangent, the sine is the least, and the tangent the greatest, by the last paragraph; therefore the difference between the sine and tangent, is greater than the difference between any other two of those four quantities. If therefore the greatest of all those differences, *viz.* that between the sine and tangent, vanishes in respect of the least of all those quantities, namely the sine; much more will the difference between any other two of those four quantities vanish in respect of the quantities themselves.

42. *Cor. 2.* Join Ba , and in the right angled triangle BFa , the hypotenuse Ba , is greater than the side Fa ; therefore $Aa - Ba$ is less than $Aa - Fa$ or AF ;

* *Archimedes De sphaera & cylindro. Defin. 2. & Axiom. 2. or Mac Laurin's Fluxions, art. 180. and 183.*

AF ; but (by par. 39.) while the arch AB decreases perpetually without limit, AF vanishes in respect of Aa ; much more then does $Aa - Ba$, the difference between Aa and Ba vanish in respect of Aa ; therefore the ultimate ratio of Aa to Ba , is that of equality; and the ultimate ratio of $\frac{1}{2}Aa$, or BC to Ba , is that of 1 to 2.

SCHOLIUM.

43. The reader will find a different proof of this celebrated proposition in *Saunderson's Fluxions*, page 248, and another in *Morgan's note on Rohault*, published by Dr. S. Clarke, Part II. Chap. 28, and re-published at *Cambridge* 1770, see page 28. The best proof of all others is undoubtedly Newton's own proof. *Princip. Lib. I. Lemma VII.* That demonstration being very concise, I shall give it here, and enlarge upon the several steps.

Let ACB (fig. 13.) be an arch of a circle, whose center is R , chord AB , tangent AD , secant RD . Let the tangent AD be produced to a distant point d , and through d draw dr parallel to the secant DR , cutting the line AR (produced indefinitely) in r ; then if with the center r , and radius Ar , an arch $Ac b$ be described, meeting dr in b ; *first*, it will be similar to the arch ACB , (though not similarly situated)*; that is, the arches $Ac b$ and ACB will have each the same proportion to the circumferences of their respective circles, because the angles $Ar d$, and ARD are equal by construction†. *Secondly*, the chord AB produced, will pass through b , and ABb will be the chord of the arch $Ac b$: for the arches $Ac b$ and ACB being similar their chords AB and Ab will make equal angles with their common tangent ADd , and therefore coincide. *Lastly*, the lines AB , AD , and arch ACB , are always proportional to the lines Ab , Ad , and arch $Ac b$.

For

* *Mac Laurin's Fluxions*, art. 122.

† 29. El. I.

For the line AB } to the { line Ab } as the radius AR , is
 the line AD } { line Ad } to the radius Ar ,
 the arch ACB } { arch Acb } or as AD is to Ad ;

therefore alternately $AB : AD :: Ab : Ad$, and likewise $AB : ACB :: Ab : Acb$: that is, the lines AB , AD , and arch ACB , are to each other, as the lines Ab , Ad , and arch Acb , are to each other. This premised, imagine now the secant RBD to turn on the center R , while the point B continually approaches to the point A . At the same time, imagine the line dr , to turn round the fixed point d , in the contrary direction, but with the same angular motion, so as always to keep parallel to RD ; by this means the center r will recede from A more and more in infinitum, and the arch Acb , will be a portion of a circle continually growing larger and larger; or as we may say, the arch Acb will continually unbend itself. In the mean time the chord ABb turning round the point A , continually increases, and as the line dr approaches to parallelism with the line Ar , the increasing chord Ab approximates to the invariable tangent Ad , and carries the intermediate arch Acb along with it. Now the point B , approaching to the point A , the angle BAD continually decreases, and there is no limit to its decrease (by prop. 16. El. III.); therefore neither is there any limit to the decrease of the angle bAd , nor to the approach of Ab to Ad ; therefore the lines Ab , Ad , and the intermediate arch Acb , will ultimately coincide and be equal; whence the chord AB , tangent AD , and arch ACB , which are to each other as Ab , Ad , and Acb , will be ultimately to each other in a ratio of equality.

We may observe, that the lines Ab , Ad , are are always finite. Ad is of a fixed invariable length; Ab approximates to Ad , can never exceed Ad , nor indeed become equal to it, 'till db is parallel to AR ; but the finite lines Ab and Ad , can approach nearer to equality, than by any assigned difference; that is, the difference

ference between Ab and Ad , vanishes in respect of those (finite) lines; whence this is inferred of the lines AB , AD always proportional to them.

Newton's artifice consists in showing us a finite part of an infinitely large circle which we *can* see; and arguing from it analogically, to an infinitely small part of a finite circle which we *can not* see.

In Newton's Lemma no mention is made of the fine; but the ultimate equality of the fine and tangent, may be easily shown in his way. From B and b , let down BF and bf , perpendicular to AR , and they will be the fines of the arches AB and Ab .

Now as the point r recedes from A , and the point b approaches to d , the line bf approximates in length to Ad , and that without limit: in other words, they are ultimately equal; whence this is inferred of the lines FB and AD , proportional to them.

Otherwise. It is evident, that as r recedes in *infinitum*, the ratio of rf to rA approximates to that of equality without limit. But $rf:rA::fb:Ad::FB:AD$; therefore the ratio of FB to AD also, or that of the fine to the tangent, approximates to equality without limit; or they are ultimately equal.

The foregoing explanation, restraining the Lemma to circles, puts a sense upon it more confined than Newton intended; but it will be the easier understood by beginners.

To propose the Lemma in more general terms, we must first premise, that the curve ACB is supposed to have continual curvature at A , and though not a circle, yet it is supposed to have circular curvature at A^* , that so the angle between the chord and tangent (while the point B approaches to A) may become less than any (assigned) right lined angle.

In the demonstration the point d is supposed to be fixed, the line Ad being of a given length. *Secondly*,
the

* *Ceterum in his omnibus supponimus curvaturam ad punctum A, nec infinite parvam esse, nec infinite magnam. Vide Scholium ad Lemma XI. Lib. I. Princip.*

the lines db and DB , are supposed to be always parallel; and the angle Adb finite. *Thirdly*, the two curves ACB and Acb , are to have one common tangent at A . *Lastly*, the arch ACB is to be always similar to the arch Acb ; that is, ACB and Acb , are to be similar parts of similar figures, the linear magnitude of Acb being to that of ACB , as Ad is to AD ; moreover, A represents corresponding points of the two figures†: for instance, if ACB be part of a circle, then must Acb be also part of a circle: the radius of Acb must be to the radius of ACB , as Ad is to AD ; and then the part Acb , cut off by the line db , and the part ACB will be similar arches of two circles. If ACB be a Parabola, then must Acb also be a Parabola, whose parameter is to the parameter of ACB , as Ad is to AD . If ACB be an Ellipse, then must Acb be an Ellipse. To make these Ellipses similar figures, the proportion of the greater axis to the less, in each Ellipse must be the same; and that the arch Acb , cut off by the line db , may be similar to the arch ACB , the greater axis of the latter Ellipse must be to the greater axis of the former, as Ad is to AD ; moreover, if the point A is the extremity of the greater axis in one Ellipse, it must likewise be so in the other.

It follows from the similarity of the arches ACB and Acb ; *First*, that their subtenses AB and Ab make equal angles with their common tangent ADd , and therefore coincide. *Secondly*, that the lines AB , AD , and arch ACB , are always to each other as the lines Ab , Ad , and arch Acb , are to each other. This premised, while the point B approaches to A , the chord Ab , turning round the point A , approximates to the invariable tangent Ad , and carries the intermediate arch Acb along with it. As the angle DAB decreases without limit, or in other words ultimately vanishes, so the angle dAb ultimately vanishes, and the finite lines Ab , Ad , and arch Acb , ultimately coinciding, are equal; therefore the evanescent lines AB , AD ,
and

† *Mac Laurin's Fluxions*, art. 122.

and arch ACB , (which are always to each other as Ab , Ad , and Acb ,) will have for their ultimate ratio, that of equality.

LEMMA I.

44. If a body be acted upon by a force, which is every where as the distance from the center A , (fig. 5.), the time of its descent to that center, will be the same from whatever place it falls. *Princip. Lib. I. Prop. XXXVIII. Cor. 2. or Keil's Physics, Th. XLV.*

LEMMA II.

45. If a body be acted upon by a force, which is every where as the distance from the center A , (fig. 5.), the time of descent through any space PA , is to the time in which it would descend through that same space if impelled by half the first force (at P) uniformly continued, as the circumference of a circle is to four diameters.

For the time of descent through the space PA , is equal to the time in which the body revolving by the force at P , in a circle whose radius is PA , would describe a quadrant, *Princip. Lib. I. Prop. XXXVIII. Cor. 1.* and in this time by the same force uniformly continued, the body would descend through a space, which is a third proportional to the diameter of the circle and quadrant thus described, *Princip. Lib. I. Prop. IV. Cor. 9.* that is calling the diameter D , and quadrant $\mathcal{Q}$, a space equal to $\frac{\mathcal{Q}\mathcal{Q}}{D}$; but the time of

descent through this space $\frac{\mathcal{Q}\mathcal{Q}}{D}$, by the uniform force at P , is to the time of descent through the space $PA = \frac{1}{2}D$, by half that uniform force, in a subduplicate ratio of those spaces directly, and a subduplicate ratio of those forces inversely*; that is, directly as $\sqrt{\frac{\mathcal{Q}\mathcal{Q}}{D}}$
is

* Introduction to *Saunderson's fluxions*. Prop. VII. Cor. IV. Can. 3.

is to $\sqrt{\frac{1}{2}}D$, and inverſely as $\sqrt{1}$ is to $\sqrt{\frac{1}{2}}$, or compounding theſe ratios, as 2 is to D , or as 4 2 is to $4D$, or as the circumference of a circle is to four diameters.

PROPOSITION I.

46. The accelerating force of a body, oſcillating in a circle, is to its abſolute weight, as the ſine of its angular diſtance from the loweſt point, is to the radius.

Let P (fig. 6.) be the body oſcillating in the arch of a circle PAp , whoſe loweſt point is A , and centre C . Let PF be the ſine of the arch PA , and let PT be a tangent to the circle at P , meeting the radius CA , produced in T , then will the accelerating force of the body at P , be the ſame as if it were placed on the tangent PT , conſidered as an inclined plane, and will therefore be to its abſolute weight, as the height of that plane is to its length *, or as FT is to PT , or becauſe of the ſimilar triangles PFT , and CFP , as PF is to CP , or as the ſine of the arch PA (the meaſure of the angular diſtance PCA) is to radius.

47. *Cor. 1.* In the ſame circle the accelerating force is every where as the ſine of the arch PA , but becauſe the ſine and arch approximate to the ratio of equality while the arch decreases, therefore the ratio of the ſines to each other will approximate to, and be ultimately equal to the ratio of their correſponding arches to each other. Wherefore ultimately, the accelerating force is as the arch PA , the diſtance of the body from the loweſt point A , meaſured along the circle in which it oſcillates.

48. *Cor. 2.* Hence the times of deſcent down unequal arches PA , eA , approximate to equality, while thoſe arches decrease in infinitum, and (as we may ſay) ultimately thoſe times will be equal. For when the accelerating force is as the diſtances PA , eA , the times of deſcent will be equal from whatever place the body begins to deſcend, (by par. 44.)

PROPO-

* Introduction to *Saunderson's Fluxions*, Prop. VII. or *Keil's* *Physics*, Lect. XV. Theorem 35.

PROPOSITION II.

49. All things as before, draw PA , the chord of the arch PeA (fig. 7.); I say that the accelerating force of the body upon the arch at P , is to the accelerating force of the body placed any where* on the chord PA , ultimately as 2 to 1.

Draw the diameter aCA , join aP and CP , let AT be a tangent to the circle at A , and PT a tangent to the circle at P , let these tangents meet in T ; then considering the chord PA , and tangent PT , as inclined planes whose heights are equal, the accelerating force on the plane PT (equal to that on the arch at P) will be to the accelerating force on the plane PA , as PA is to PT †, or because of the similar triangles PAT and PaC ‡, as Pa is to PC , that is ultimately as 2 is to 1 ||.

50. *Cor. 1.* The time of descent through the arch PeA , is to the time of descent through the chord PA , ultimately, as the circumference is to four diameters. For ultimately, the length of the arch and chord are equal, and ultimately, the accelerating force of the body on the arch is every where as its distance from the lowest point A , (by par. 47.) but the accelerating force of the body on the chord is uniform and equal to half the first force on the arch at P , by this proposition, therefore (by par. 45.) the time of descent through the arch, is to the time of descent through the chord, as the circumference is to four diameters.

51. *Cor. 2.* The time of one oscillation is to the time of descent down half the length of the pendulum ultimately, as the circumference of a circle to its diameter. For the time of descent through the chord PA ,

* Introduction to *Saunderson's Fluxions*, Prop. VII. Cor. I. or *Keil's Physics*, Lect. XV. Theorem 35.

† Introduction to *Saunderson's Fluxions*, Prop. VII. Cor. I. or *Keil's Physics*, Lect. XV. Theorem 35.

‡ 32. El. III.

|| Par. 42.

PA , being always equal to the time of descent down the diameter aA , we have *ultimately*, the time of descent through the arch, to the time of descent down the diameter, as the circumference is to four diameters by the last corollary; whence the descent and ascent along the arch, or the time of one oscillation, is to the time of descent down the diameter, as the circumference to two diameters; but the descent down the diameter, is to the descent down $\frac{1}{4}$ of the diameter, or half the length of the pendulum, as 2 to 1; therefore (compounding these ratios) the time of one oscillation, is to the time of descent down half the length of the pendulum, as the circumference of a circle to its diameter.

SCHOLIUM.

52. If the sines increased in the same ratio with the arches or distances PeA , all oscillations would be precisely equal; but as the sines increase in a less ratio than their arches, the accelerating force, when the arch is enlarged, is not sufficiently increased to make the times of descent, and consequently the oscillations equal; therefore the oscillations through larger arches take up a longer time. Contrariwise, if the arch be lessened, the time of descent is lessened, and therefore the time of descent and ascent, or the time of one oscillation is lessened. But though the arch may decrease in infinitum without any limit, yet the time of an oscillation through that arch will not decrease proportionably, nor in infinitum, but has a limit. Thus, if the length of the pendulum be 39,2 inches, the time of one oscillation through any arch will be somewhat more than one second. Now though that arch may be taken less than any assigned arch, yet the time of one oscillation will not become less than any assigned time. For if the pendulum oscillates at all, it must take up a determinate finite time, namely one second,

§ Keil's Physics, Lect. XV. Theorem 39.

second, how small soever the arch may be, through which it oscillates. Thus then, although strictly speaking, there is no such thing as the time of a *least oscillation*, (there being no *least arch*, and therefore no least oscillation;) yet there is, strictly speaking, a *least time* of the oscillation of every pendulum; and the proportion of this *least time* to the time of descent, down half the length of that pendulum, has been before pointed out.

53. The time here sought, is that which is the limit of the varying time of an oscillation, through an arch supposed to decrease continually without limit; therefore, in comparing those quantities on which the time of one oscillation depends, it is the limit of their varying ratio, or rather the limit of the varying ratio of the length of those lines, to which such quantities are analogous, that must be attended to*.

In the first corollary to the first proposition (par. 47.) when we had found the accelerating force to be always as the sines, we thence concluded it to be ultimately as the arches, because of the ultimate equality of the sine and arch. In the first corollary to the second proposition (par. 50.) we compared the time of descent along the arch, with the time of descent along the chord, and as far as those times depend on the length of the line described, (and not on the accelerating force) we considered them as ultimately equal, because of the ultimate equality of the length of the arch and chord. In both these cases then we admitted this proposition concerning the ultimate equality of the sine, chord, and arch; and it was rightly applied, because the ratio of the quantities to be compared, (the forces in the former case, par. 47. and the times in the latter, par. 50.) depended on the ratio of the *length* of these lines; but we misunderstand the proposition,

* It may be observed, that in this and similar cases, the very nature of the problem shows that it is the limit of the varying quantity, or the limit of the varying ratio, that we are to seek; as will be further exemplified in par. 63.

position, if we suppose it to imply that the arch and chord do ultimately *so coincide* that they may be considered as *one and the same line*. For on whatever grounds we suppose the arch thus to coincide with the chord, we might likewise suppose it to coincide with the sine PF , or with the tangent to the lowest point TA (fig. 8.) for these are all *ultimately equal*; but both sine and tangent in this case are parallel to the horizon, and therefore there would be no accelerating force, no oscillation at all; and thus the whole enquiry would fall to the ground. On the contrary; the arch is a curve, all whose parts have different inclinations to the horizon, but the chord is a right line which has every where the same inclination. The accelerating force on the arch continually increases with the distance from the lowest point, but the force on the chord is every where the same. Neither the law of the accelerating force, nor the quantity of it at the beginning of the descent in these two cases approximate by the decrease of the arch; we have therefore no reason to expect that the times of descent should approximate, notwithstanding the lengths of the lines descended through, do approximate and become ultimately equal. *Keil*, and others after him, say, that the *arch and chord differ but little either in length or declivity, therefore neither do the times of descent**. It should be shown, that the times are in some finite ratio of these quantities, otherwise, small differences in these circumstances may occasion great differences in the times of descent. The assertion (rightly explained) is true with regard to the length of the chord and arch, but not with respect to their declivity, if we allow that any comparison can be made between the declivity of a curve and a right line, which yet seems to be absurd.

54. The ratio of the length of the arch to the sine, or of the arch to the chord, which varies while the arch decreases,

* *Keil's Physics*, Lect. XV. Theorem 41. *Monf. Parent*, I believe was the first that fell into this error.

decreases, *has for its limit the ratio of equality*, the difference of their lengths vanishing in respect of the lengths themselves, all which is only expressed in other words, when we say that they are *ultimately equal*; but the arch is ever to be considered as a CURVE, and the chord as a RIGHT LINE, two ideas which ought not to be confounded.

55. Newton says — *figura rectilinea, quæ chordis evanescentium arcuum comprehenditur, coincidit ultimo cum figura curvilinea. Princip. Lib. 1. Lemma III. Cor. 2.* — This is the express language of indivisibles, and must not be understood literally, any more than *summa ultima* in the preceding corollary. Newton himself expounds *summa ultima*, and *ratio ultima*, by *limites summorum & rationum*, and rests the whole force of his demonstrations upon the doctrine of LIMITS. *Scholium ad Lem. XI. pag. 37, Ed. 3.*

56. Another example of thus speaking in the language of indivisibles, is in the fourth corollary of the same *Lemma III.* *Figura ultima*, is an expression plainly absurd if taken literally. It supposes a transformation of a right lined figure into a curvilinear figure. But surely that figure is always a right lined figure, which you affirm is always bounded by right lines, be their number ever so great. Newton cautions his reader, that what he says, respects only the perimeter, that is, the length of the perimeter of the figure. This corollary is quoted once only, Book I. Prop. 1. and in the way it is there applied relates to a comparison, either of the length of the perimeter, of the inscribed rectilinear figure, with the perimeter of the circumscribed curve; or to a comparison of the area of the rectilinear figure, with the area of the curve; but has no respect whatever to the CURVATURE, or form of the curve.

57. The like objection may be made to Lemma VIII. There is no *ultima forma*. Nor can the shape of a figure bounded by two right lines and a curve, become the same, or similar, to one bounded by three right
right

right lines. It is true, the ratio of the length of two lines, of whatever shape, or the ratio of two areas, however bounded, may either be, or become, that of equality; which is all that is here intended.

58. The equality of the three triangles in Lemma VIII. viz. RAB , $RACB$, RAD (fig. 14.) (all that is of importance) may be proved in a short way thus: The triangle RAB is to the triangle RAD , as BF to DA ; that is, ultimately in a ratio of equality, as has been shown (par. 43.): of course $RACB$, which is of an intermediate magnitude, will be ultimately equal to the other two triangles.

59. We may further observe, that when the angle bAd vanishes, the angle ArD (the double of bAd) must vanish too, but the angles rAd and rdA are always finite; therefore in the triangle ArD , the ratio of the side Ar , to the side Ad , increases without limit; and ultimately, either the side Ar must become *infinitely* great, or the side Ad *infinitely* small; of course, the triangles rAb , rAc , rAd , are *not always finite*, and this, whatever be the supposed motion of the line rd .

A triangle, having two evanescent angles, may have all its sides of a finite magnitude; but not if it has only one evanescent angle.

60. Newton frequently expresses himself in such terms as were used by the writers on indivisibles in his time; and though at the end of that remarkable scholium, he cautions his reader against being misled by the use of those terms, yet he seems sometimes to forget that caution himself, of which these lemma's and their corollaries are an instance. The very expression — *ultimæ quantitatum evanescentium rationes*, (or ultimate ratios), taken literally, implies the ratios of quantities in an ultimate state, that is, of quantities infinitely small, or infinitely great: and though Newton afterwards says, that by ultimate ratios, is meant the limit of the varying ratio of decreasing quantities, and not the ratio of ultimate quantities, yet he still speaks of varying ratios, as reaching their limit —

quum quantitates diminuuntur in infinitum, — when the terms become infinitely little.

61. It will be said, that all this is only to cavil about words; — so it is, with those who interpret rightly this mode of speaking; but it is what will certainly mislead common readers, and almost as certainly prove a snare to the very ablest writer, if he is not continually upon his guard.

62. We said before, that in problems of this sort, their nature shows, that it is the limit of some varying ratio that we are to seek, and not the ratio itself. Thus, if the areas of two curvilinear spaces are to be compared together, let their bases be each divided into the same number of parts, and on those divisions let parallelograms be drawn, so as to form right lined figures, inscribed within the curvilinear ones, as is directed in the fourth lemma of the first book of the *Principia*. Now, both the ratio of these parallelograms (each to each) and of the right lined figures composed of them, all will vary, as the number of parts varies into which the base of each figure is divided, and the limit of the varying ratio (when it can be found) is that of the curvilinear figures themselves, as is proved in the lemma aforesaid; and a very neat example of finding that limit in a particular case, is given by *Saunderson* in his *Fluxions*, page 243. This, with some alterations, we shall here insert.

PROBLEM.

63. Let AM (fig. 16.) be a Parabola, whose axis is AP , parameter t . Let PM and RN be ordinates to the axis, meeting the curve in M and N . It is required to find the parabolic space AMP .

Complete the parallelograms $APMQ$ and $ARNT$. Let TN and RN produced, meet PM and MQ , in U and S respectively. Then $t \times AP = PM^2$, and $t \times AR = RN^2$; therefore $t \times PR = PM^2 - RN^2 = RS^2 - RN^2$

$RN^2 = \overline{RS + RN} \times \overline{RS - RN} = \overline{RS + RN} \times \overline{NS}$; that is, $t \times PR = \overline{RS + RN} \times \overline{NS}$, and $t \times AR \times PR = \overline{RS + RN} \times \overline{AR} \times \overline{NS}$, or $RN^2 \times PR = \overline{RS + RN} \times \overline{AR} \times \overline{NS}$: but $RN \times PR$, is equal to the rectangular parallelogram NP , and $RN^2 \times PR$ is equal to the parallelogram NP , drawn into, (or multiplied by RN . In like manner, $\overline{RS + RN} \times \overline{AR} \times \overline{NS} = \overline{RS + RN} \times \overline{TN} \times \overline{NS}$ is equal to the parallelogram NQ , drawn into $\overline{RS + RN}$; and the parallelogram NP , is to the parallelogram NQ , as $RS + RN$ is to RN .

Suppose now the axis AP , to be divided into many parts (equal or unequal) and through each point of division draw ordinates as NR , and complete the several parallelograms as before; forming in the whole two rectilinear figures, the former inscribed in the curvilinear space AMP , the latter in the curvilinear space AMQ ; the number of little parallelograms in each figure being equal, as in Lemma IV. *Principia*. Then, by what has been shown, any two corresponding parallelograms will be to each other, as $RS + RN$ to RN . Increase now the number of these parallelograms, diminishing their breadth, and the ratio of $RS + RN$ to RN , will approximate to, and ultimately terminate in the ratio of 2 to 1; and the ultimate ratio of any two corresponding parallelograms will be that of 2 to 1. Hence, the *ultimate sum* of all the parallelograms inscribed within AMP (when their number is increased, and their breadth is diminished *in infinitum*) will be to the like ultimate sum of all the parallelograms inscribed within AMQ , as 2 to 1; or the ultimate ratio of the whole rectilinear figure inscribed within AMP , to the whole rectilinear figure inscribed within AMQ , is that of 2 to 1. But the ultimate ratio of these inscribed rectilinear figures, is that of their circumscribing curves: therefore the internal space AMP is double the external space AMQ ; or the

the parabolic space AMP , two thirds of the circumscribing parallelogram.

64. REMARK. The nature of the problem, (*viz.* to find the ratio of the curvilinear figures) shows plainly that it is not the ratio of the inscribed rectilinear figures that we are to seek, how great soever the number of the little parallelograms that compose them may be, but the LIMIT of that (varying) ratio.

65. It may be worth while to illustrate this by another instance in the solution of the following

PROBLEM.

To find the proportion of the velocities of two flowing lines, at a given instant, from their known cotemporary increments.

Let us for the present suppose the velocities of these lines to be continually accelerated; now if the time in which their cotemporary increments are generated, be supposed perpetually to decrease, the ratio of those increments will continually vary; and the limit of their varying ratio is that of the velocities sought.

For every such increment may be distinguished into two parts; *first*, that which generated by the velocity of the flowing line at the given instant, continued uniformly for the whole time in which the increment is acquired; and, *secondly*, that part of it which is generated in consequence of the subsequent increase of the first velocity. Now this additional velocity grows gradually from nothing, therefore by lessening the time in which the increment is produced, this additional velocity will continually lessen, and will become less than any assigned fractional part of the first velocity, which the line had at the given instant. Hence, the latter part of the increment, generated in consequence of the additional velocity, will ultimately vanish in respect of the former part of the increment,
generated

generated by the first velocity immutably continued, even though by thus diminishing the time, the former part of the increment does itself vanish in respect of any assigned or finite line. In any such flowing line then, if the whole increment be compared with that part of it which is generated by the first velocity uniformly continued, *first*, these two quantities continually approximate to each other, as the time in which they are generated decreases; *secondly*, the one can never pass beyond the other, for the whole increment can never become less than a part of it: *lastly*, their difference, namely, the part generated in consequence of the acceleration, vanishes in respect of the quantities themselves; therefore (by par. 26.) the limit of their varying ratio is that of equality.

In each line then, the whole increment is to the former part of it (generated by the velocity at the given instant uniformly continued) ultimately in a ratio of equality, and alternately, the whole increment of one flowing line, is to the whole cotemporary increment of the other flowing line, ultimately, as the former part of the increment of the first flowing line, is to the former part of the increment of the other flowing line; that is, as the velocity of the first flowing line at the given instant, is to the velocity of the other flowing line.

66. Here the nature of the problem directs us to seek the *limit* of the varying ratio of the cotemporary increments of these lines, (varying as the time in which they are generated decreases perpetually without limit), not to seek the ratio of the increments themselves, how small soever we may suppose the time to be, in which they are generated.

67. Having shown this in general, we may now proceed to enquire, what the limit of that varying ratio is in any particular given instance. For example, let the former of these two lines be called x , and let the latter line be a third proportional to a fixed line (assumed at pleasure) called unity, and to the former

mer line x , then will its value be xx ; that is, x and xx will represent any two cotemporary values of lines, having this particular relation to each other. Let v be an increment of the line x , then will $x+v$ and $(x+v)^2 = xx + 2xv + vv$, be two other cotemporary values of these flowing lines, whence their cotemporary increments will be v , and $2xv + vv$, whose ratio is that of 1 to $2x+v$. Now, supposing the time in which the increment v is generated, to decrease continually without limit, the value of v will likewise decrease without limit, and will ultimately vanish in respect of the finite line $2x$, which is not affected by lessening the increment v , so that $2x+v$ will be to $2x$, ultimately in a ratio of equality. Hence, the limit of the varying ratio of 1 to $2x+v$, while v decreases without limit, will be that of 1 to $2x$, which is therefore the proportion of the velocities sought: therefore it will be, as the fixed line, called unity, is to twice the flowing line x , so is the velocity of the line x , to the cotemporary velocity of the line xx .

68. The whole argument will be the same (*mutatis mutandis*) if we suppose the velocities of these lines to be continually retarded. The increment may in that case be distinguished into two parts, that which would be generated by the first velocity uniformly continued, and that which is lost by the retardation. Now, when the time is continually diminished, this latter part will ultimately vanish in respect of the former part, as before. If we suppose the velocities of the flowing lines to be uniform, then their cotemporary increments are the measures of those velocities.

69. *Cor.* If the velocity of the line x be represented by $\dot{x}$, the cotemporary velocity of the line xx , will be represented by $2x\dot{x}$. For 1 is to $2x$, as $\dot{x}$ is to $2x\dot{x}$.

CONCLU-

CONCLUSION.

70. *Mac Laurin* professes to demonstrate the principles of fluxions after the manner of the antient Geometricians. (See his Fluxions, page 51.). By this means the truth of Newton's propositions is shown beyond a doubt; but surely this is no defence of Newton's method of reasoning, so much impeached by the author of the *Analyst*. If *Mac Laurin's* way of demonstrating the grounds of the method of fluxions has all the accuracy, it has likewise all the tediousness of the antients. Every proposition in his first book is divided into two cases, each of which he is obliged to demonstrate separately. It was to avoid this very thing, that Newton introduced a new way of reasoning on these subjects: that it is just and logical, is the point we have endeavoured to make out in this essay.

ESSAY

ESSAY II.

ON THE POWER OF THE WEDGE.

1. **L**ET PEH (fig. 9.) be a rectangular wedge; whose angular point is E , face PE , base EH , and back PH . Let this wedge slide freely on the plane LN . Let a body E be urged in the direction KE , against the face of the wedge, and let that body be kept in the direction KE , either by bearing against a plane parallel to KE , or by a string EB , always drawing the body at right angles to KE ; or let the body be in any manner prevented from moving along PE , and compelled to move in the line KE , by a force acting at right angles to KE .

2. Given the angle of the wedge PEH , and the angle KEP , which the direction of the force upon E , makes with the face of the wedge: to find the proportion between the force applied to the body in the direction KE , and the force applied perpendicularly to the back of the wedge, when these two forces balance each other, so that the wedge moves neither forward nor backward. To find likewise the proportion of the forces aforesaid, to the pressure of the wedge, against the plane LN , and to the pressure of the body E , against the plane parallel to KE , occasioned by the action of the other two forces, when they balance each other.

Before we enter upon the solution of the problem, it may be proper to premise the following mechanical principles, universally received.

3. *First*, When a body is acted upon by one force only, it cannot be sustained by another single force, unless the latter be both equal and *opposite* to the former.

4. *Secondly*, When two or more forces act at once upon a body, the combined effect of them all, is to move it in one direction only; and they will have the same effect, and be equivalent to a single force, acting in one particular direction*.

5. *Thirdly*, When an hard body acts *in any direction whatever* on the surface of another, and such surface is perfectly hard and smooth; whether the former body presses continually on the latter, or only strikes it, yet the effect of that pressure in one case, or of the impact in the other, and the consequent motion in the body *acted upon* (supposing it at liberty,) is always in a DIRECTION PERPENDICULAR TO THAT SURFACE†.

6. These principles laid down, it is manifest, that when the body *E* is both urged in the direction *KE*, and at the same time sustained in the line *KE*, by a plane parallel to *KE*, the original force in the direction *KE*, combined with the reaction of the sustaining plane (or with the force of the string *EB* drawing at right angles to *KE*) may be considered as a single force, urging the body *E* against the face of the wedge:

* The rule for finding the quantity of that force, and the particular direction in which it must act, is in almost every book of mechanics. See Essay III.

† When a plane perfectly hard and smooth is sustained against the pressure of an hard body, it is said to *react* against the body that presses it, and as the sustaining force must be in a direction opposite to that in which the plane would move if at liberty, the reaction of the plane is therefore perpendicular to its surface.

The force that sustains such a plane must be equal to the force of the body that presses it, that is, the reaction of the plane against the body, is equal to the action of the body upon the plane.

The principle here laid down is further explained in par. 21.

wedge: but the action of an hard body E , against the smooth and hard surface of the wedge is always perpendicular to its face PE , and a single force impressed on the wedge in the opposite direction, *and in that only*, will counteract the other two (combined) forces and keep the wedge in equilibrio. Now, the directions of these three forces being known, their proportions, when they balance each other, are easily found by a common rule in mechanics. For through P draw PA , parallel to KE , through E draw ED , perpendicular to KE , meeting PA in D , also draw EA perpendicular to PE , meeting PA in A ; and ED , DA , AE , the sides of the triangle EDA , lying in the direction of the forces aforesaid, are to each other as those forces †.

7. EA

† It may perhaps throw some light upon this subject if we consider the whole effect of the original force in the direction KE , and the whole effect of the reaction of the sustaining plane, separately from each other.

Now, when a force acts upon the body E , in a direction KE , oblique to the hard and smooth surface PE , such a force will produce in the body E a tendency to move along that surface PE , (down the face of the wedge), and likewise a pressure of the body E against the wedge, perpendicular to its face PE . The body E is here supposed to be kept from moving down the face of the wedge by another force, namely, the reaction of the sustaining plane: this force is perpendicular to the former force KE , therefore likewise oblique to the surface PE . This force then, being in the oblique direction ED , partly urges the body E up the face of the wedge, and partly adds to the perpendicular pressure of the body E , against the wedge. From D let fall Dd , perpendicular to EA ; and if DA , as before, represents the original force applied to the body E in the direction KE , then will Dd be the force urging the body down PE , and dA the force urging the body against the wedge. Again, if the sustaining plane prevents the body from moving down the face of the wedge, that part of the reaction of this plane which is in the direction EP , must be equal and contrary to Dd , and therefore is represented by dD ; whence, ED will represent the whole reacting force of that plane, and Ed will represent the other part of this force, which urges the body perpendicularly against the face of the wedge. To Ed add dA , and their sum EA , will be the whole force with which the body E presses the face of the wedge, as before.

For the nature and laws of oblique forces, see par. 21.

7. EA then represents the quantity and direction of the force impressed on the wedge by the body E , but the effort of the wedge to move in the direction EA , is not in our case sustained by a single force equal and opposite, but by two others, one the reaction of the plane on which the wedge slides, and the other the force on the back. The force EA , being partly opposed to the plane LN , and partly to the back PH , will produce partly a pressure of the wedge upon the plane LN , which is balanced by the reaction of that plane, and partly a tendency in the wedge to move backwards, which is balanced by the force applied to the back. Now, the directions of these two forces, as well as that of EA , are known, whence their proportions when they balance each other will be thus found. From A draw AF , perpendicular to EH ; and EA representing as before the quantity and direction of the force impressed on the wedge; FA will represent the pressure on the plane LN , and FE the force to be applied perpendicularly to the back of the wedge, to keep it in equilibrio.

8. If then DA represents the force applied to the body E , in the direction KE ; FE will be the force applied perpendicularly to the back of the wedge, when these two forces balance each other: FA is the pressure of the wedge upon the plane EH , and DE the pressure of the body against the plane, parallel to KE (or it is the tension of the string EB) occasioned by the action of the two former forces, when they balance each other.

9. The right angled triangles AED , AEF , having one common hypotenuse AE , are similar to the right angled triangles EPD , EPH , having one common hypotenuse EP ; the sides of the former being respectively perpendicular to the sides of the latter. Therefore DA , FE , FA , and DE , are respectively as ED , PH , EH , and PD , or making EP radius, as the sine of $EPD=KEP$; the sine of PEH ; the cosine of PEH ; and the cosine of EPD , or KEP ; so that

D

the

the force applied to the body in the direction KE , is to that on the back of the wedge which balances it; as the sine of KEP , the angle of direction of the force upon E , to the sine of PEH , the angle of the wedge: the pressure on the plane LN , is represented by the cosine of the latter angle, and the pressure against the plane parallel to KE , by the cosine of the former angle.

10. *Cor. 1.* Let PA cut EH in I , and draw PK parallel to EH , meeting EK in K , and while the wedge moves upon the plane NL , in the direction NL , or PK , through the space $PK=IE$, the body E will be raised in its proper direction through $EK=PI$; therefore their respective cotemporary velocities, estimated in the directions in which the forces on each are respectively applied, are as EI to PI , or because of the similar triangles EDI , PIH , as ED to PH , or reciprocally as those forces, according to the rule given in par. 20.

11. *Cor. 2.* When KE is perpendicular to PE , the point D coincides with P , and $ED=EP$, and PD is nothing; that is, the pressure against the sustaining plane (as we called it) is nothing. The other three forces are as EP , PH , and HE , the sides of the wedge to which they are respectively at right angles. This is the case commonly considered by writers on mechanics.

12. *Cor. 3.* When KE is parallel to PH , the points D and H coincide, and $ED=EH$, and the force on the body in the direction KE , is to that on the back of the wedge which balances it, as EH is to PH . PD is likewise equal to PH , that is, the pressure against the sustaining plane is equal to the force applied to the back of the wedge.

13. *Cor. 4.* If the wedge be in the form of an isosceles triangle ABC , (fig. 10.) whose sides are AC and BC , back AB , perpendicular height CH , and be applied to separate two bodies, E and e , in directions EF , and ef parallel to the back, but opposite to each other,

other, which bodies are kept in their respective directions, by bearing against a plane, or by a string in the manner before laid down; then when the wedge is in equilibrio, the perpendicular force on the back, will be to each of the (equal) forces FE , or fe on the sides, as the whole back AB is to CH , the perpendicular height of the wedge.

This will appear, by considering such a wedge as composed of two rectangular ones, ACH and BCH . In this case, the pressure of the two wedges ACH and BCH , against one another (in the directions AH and BH) being equal and opposite, will destroy each others effect. Now FE , the force on the side AC , will be balanced by a perpendicular force on the back, which is to FE as AH is to CH , (by par. 12.) and another perpendicular force on the back, equal to the former, will balance fe , the force on the other side BC . Therefore the whole perpendicular force on the back, which balances both forces on the sides, is to one only of those forces, as $2AH$ or AB is to CH .

SCHOLIUM I.

14. The case last described is exactly that of the machines, generally made use of to shew experimentally the power of the wedge. In that described by *'s Gravesande* in the first and second editions of his *Experimental Philosophy*; by *Hawksbee* and *Whiston* in their *Course of Lectures*; by *Desaguliers* in his *course of Experimental Philosophy*, Lect. 3. par. 58. in all these the rollers to be separated by the wedge, rest upon the edge of a brass frame, which determines the direction in which they must move.

These rollers are drawn together by weights tied to strings, which, passing over a pulley, run along the edge of the frame, and therefore the weights act upon the rollers in the direction of that edge. The brass frame being set level, the wedge is put between the rollers, its back being parallel to the frame, and

to the horizon. The wedge is then laden with weight till it begins to separate the rollers, which will be when the weight of the wedge itself, and its load, is to the weight with which each roller *separately* is drawn along the frame, as the back of the wedge is to its perpendicular height.

15. In the machine described by *'s Gravesande*, in his third edition, N^o. 279. and by Mr. *Ferguson* in his Lectures, page 68. Ed. 1760. the rollers, instead of resting upon the horizontal edge of a brass frame, are suspended by long lines, which not only support the weight of these rollers, but also determine the direction in which they must move when separated, namely, at right angles to these lines; that is, in a direction parallel to the horizon, as in the former machine.

16. It seems not to be observed, that when the wedge is put between the rollers, the reaction of the brass plane on which they rest, in the former machine, and the tension of the suspending lines in the latter, is increased by the whole weight of the wedge and its load. This would appear very plain in the latter machine, if the lines by which the rollers are suspended, passed over pulleys instead of being fixed to hooks in the ceiling; for the rollers might then be balanced so as to hang suspended in equilibrio before the wedge is put between them. In such a case, when the wedge is put in, that equilibrium will be destroyed, and to restore it, the rollers must be drawn upwards by the addition of a weight, equal to the wedge, and its load.

17. It therefore appears plainly, that in making this experiment, the rollers are not only drawn horizontally, but both of them upwards; that the rollers are each of them acted upon, not by one oblique force only, as these authors suppose, but by two; that the action of the wedge against the rollers (when they thus balance each other) does *of necessity* create an additional tension of the fixed lines in *'s Gravesande's* latter experiment, and an additional pressure upon the
brass

brass plane in his former experiment; and lastly, that the quantity of this additional action of the rollers upon the brass plane, and its consequent reaction, is exactly what we determined it to be in par. 12.

18. The two forces, which thus act upon each roller, are both of them oblique to the side of the wedge, but lie on opposite sides of the perpendicular, their directions being at right angles to each other. Now, these two forces will of necessity ASSUME such a proportion, that the force compounded of them both, shall be perpendicular to the side of the wedge, otherwise there could be no equilibrium. For the force compounded of them both is sustained by another single force, namely, the reaction of the side of the wedge, whose direction is at right angles to this surface, and upon this principle, the proportion of these three forces was determined in par. 6.* Thus then the direction in which the rollers act against the wedge, is always perpendicular to its sides, however it may seem otherwise. A machine may be made in which an equilibrium shall be produced, when TWO forces act upon each roller, in directions oblique to the sides of the wedge; because the rollers may in this case be urged perpendicularly against those sides; but no machine can be made in which *all* the parts shall be at rest, when ONE force only acts on each roller, in a direction oblique to the side of the wedge. See par. 21. and 29.

19. Whilst we are upon this subject, it may not be amiss to take notice of one or two other circumstances in these experiments, which have occasioned much needless disputation. In *Desaguliers's* machine, and all others, but that described by *'s Gravesande* in his third edition, the strings which draw the rollers together run over fixed pulleys; that is, fixed to the wall of the room, or to some firm body, not connected with the rollers. Each weight then shows the force acting upon the roller it draws along, and the sum of the

* See par. 3. 4. and 5.

the two weights is the sum of the two forces, acting upon the rollers. In the machine described in 's *Gravesande's* third edition, a pulley is suspended by a long line from the ceiling; this pulley is fastened by an horizontal line to one of the moveable rollers, and a string being tied to the other roller, is put over that pulley, at the end of which is hung a single weight, by which both rollers are drawn together horizontally. In this last case, each roller is acted upon by the whole of that single weight, and the sum of the two forces, acting upon the two rollers, is twice that single weight. Now, what shall we understand by the force with which the rollers *cohere*? or with which they are *drawn together*? or what shall we understand by the *resistance* to be overcome? shall we understand by it, the force with which each roller singly is drawn along the brass frame? or shall we understand by it, the sum of the two forces by which the two rollers are drawn along the brass frame in opposite directions, and which is double the former force? We have shown in par. 13. that in the former sense of the words, the force on the back of the wedge, (or as it is often called, the *power*) is to the *resistance*, as the whole back of the wedge is to its perpendicular height; and therefore taking *resistance* in the latter sense, the *power* will be to the *resistance* as the whole back to twice the perpendicular height, or as *half the back to the perpendicular height*. Now this *whole* resistance in the machine described by *Desaguliers*, where each roller is acted upon by separate weights, is the sum of those two (equal) weights: therefore there will be an equilibrium when the weight of the wedge and its load, is to the sum of those two weights, as half the back to the perpendicular height. But in 's *Gravesande's* last machine, the *whole* resistance is twice the single weight by which both rollers are drawn together, as was before observed: therefore there will be an equilibrium when the weight of the wedge and its load, is to that single weight, as the whole back of the wedge,

wedge, to the perpendicular height. And thus the effect of one weight, acting at once on both rollers in *'s Gravesande's* machine, is equivalent to that of two weights each equal to that one, but acting separately (one upon each roller) in *Desaguliers's* machine.

In all these cases, the difficulty is only about *words*; by what names we should, or rather by what names we *do*, usually call things; and if the language held on these occasions be not settled by an uniform use of the terms among writers, no wonder that confusion should follow. Settle but the meaning of the terms, and all disputing vanishes; but dealers in this sort of philosophy, not suspecting that they are talking to one another in a different language, *make experiments* to prove, as they say, the truth of their propositions. The experiments on both sides always succeed, and to our surprise, the controversy is no nearer to an end. — Why? — Because these philosophers are making experiments to find — *the meaning of a word**.

SCHOLIUM II.

20. Most of the writers on mechanics, in demonstrating the power of the wedge, make use of the rule mentioned by Newton, “*that in mechanical engines, those powers balance each other, which are reciprocally as the velocities of the parts of the engines to which they are applied, estimated according to the respective directions in which the powers act*†.” Newton takes notice of the analogy of this rule to the rules concerning the collision of bodies, but gives no proof of it; nor does *Mac Laurin*,

* The dispute about the FORCE OF BODIES IN MOTION, in which all the philosophers in Europe were engaged for half a century together, was undoubtedly of this sort. See *An Essay on Quantity* by Mr. Reid, in the Philosophical Transactions, N^o. 489. Vol. XLV. for the year 1748, or *Martyn's* Abridgement, Vol. X. Part I. page 22. This little essay, hitherto unnoticed, well deserve the attention of every one who would understand either mathematical or philosophical subjects clearly.

† *Principia*. Cor. VI. ad Leg. mot. pag. 26. Ed. 3.

Laurin, who mentions it from *Newton*, attempt to demonstrate it †. *Newton* himself demonstrates the fundamental proposition about the lever, by the resolution of forces; a doctrine ultimately referred by him to the first and second laws of motion, which laws he lays down as first principles or general axioms, from which all particular kinds of motion are to be mathematically derived*. The best writers following *Newton*'s example, demonstrate the other mechanical powers from the same principles †, but in treating of the wedge, always restrain their proof to a particular case where the forces are applied at right angles to the sides of the wedge. It may be said, perhaps, that three forces only (one to each side) cannot be otherwise applied so as to make an equilibrium. Let us see however what determination the theory gives in one

† *Mac Laurin*'s account, &c. Book II. Chap. 3. Sect. 2. Also Book II. Chap. 2. Sect. 19. and 23. *Mac Laurin* seems to think this Analogy or *Harmony* as he calls it, a sufficient proof of the truth of this rule.

* Or as we call it *solved*. To solve a phenomenon, is only to derive it from, or show that it is included in, the general rules of motion. Thus we reduce that vast variety of motions we see in the world, into a few general classes, and show that the universe, thus regulated by a few simple laws, is the effect of supreme intelligence. To solve a phenomenon then, is not to assign its cause. When we say that gravity is the cause of the moon's motion, we mean only that its motion is of the same sort with that of other bodies near the earth; that the moon's motion is conformable in its circumstances to certain laws of motion which obtain universally; as to the cause, for aught we know, an angel may carry it about.

Newton sufficiently warns his readers, (if they will take any warning) against this common conceit that Philosophy has to do with causes. See *Principia*. Def. VIII. at the end. Page 6. Ed. 3. Want of attention to this, has introduced a great deal of absurd metaphysics into natural philosophy, of which the reader may see enough in a paper by Dr. S. Clarke, On the Force of Bodies in Motion. *Philos. Trans.* Nr. 401.

† *Mac Laurin*'s account, &c. *Morgan*'s notes on *Robault*, republished at Cambridge, 1770.

The futility of the common proof, drawn from what is called the equality of *momentums*, is sufficiently shown by Dr. *Hamilton* in his ingenious Essay on the Principles of Mechanics.

one of the simplest cases of oblique forces, namely, when two equal forces act on the sides of an isosceles wedge, in directions parallel to the back, but opposite to each other, and are sustained by a third force acting perpendicularly on the back of the wedge*.

21. Before we enter upon the solution of this problem, it may be proper to premise an account of the nature and laws of these oblique forces.

Strictly speaking, there is no such thing in the abstract as an oblique force, or a force producing a motion in a *single body*, not in the direction of that force. The motion produced in every body is supposed to be in the direction of the force that produces it. Indeed, the direction of the motion produced, is what marks out the direction of the force producing it, and the quantity of this motion is the measure of that force†. But one force only may produce a motion in *two* bodies (acting against each other) neither of which motions, *taken separately*, shall be in the direction of the original force producing them. Thus one body may be urged against (or may strike) the surface of another, in a direction oblique to that surface, and then (supposing them hard) the motion produced in each of these *two* bodies (considered separately) will neither be in the direction of the original force, nor will its quantity be so great as what the original force alone would produce. To determine the effect of such an oblique action of one body against another, the original force is to be resolved into two, one parallel

* Dr. *Hamilton* in his *Essay*, page 165, proposes to consider the case of oblique forces, but his demonstration, if conclusive, respects a very different one. In this demonstration the *protruding* force *EP* (fig. 11.) is considered as equivalent to two others *GP* and *EG*, that is, the perpendicular reaction of the side of the wedge against the body *E*, will be sustained by the joint action of the two oblique forces *GP* and *EG* upon the body: True — but then we suppose the body *E* to be sustained not by one, but by two forces, both of them (separately) oblique to the side of the wedge, but conjointly perpendicular; both necessary to make an equilibrium, therefore neither can properly be said to *be lost*.

† See *Essay* III.

parallel and one perpendicular to the surface of the body acted upon. The separate effect of these two forces will thus be manifest, and by the laws of motion, the effect of two such forces, conjointly, is the same as that of the original force alone. Now the quantity and direction of the motion, produced in *the body acting against the plane*, is such as would be produced by the parallel force alone. The perpendicular force indeed, urges the body against the plane (whose reaction sustains that force) but creates no motion in this body along the plane*. Again; the quantity and direction of the motion, impressed on *the body whose surface is acted upon*, is such as would be produced by the perpendicular force alone, the parallel force creating no pressure of the *acting body* against that plane. Having thus laid down the rule for finding the direction of the forces, impressed on these bodies separately, and their proportion to the original force, we proceed to to the solution of the problem.

22. Let ABC (fig. 10.) be the isosceles wedge, whose angular point is C , sides AC and BC , back AB , perpendicular height HC . Let FE represent the quantity and direction of the force, applied to one of the sides; resolve this into two other forces, FD and DE , the former parallel, the latter perpendicular to the side AC , and the original oblique force FE , will have just the same effect upon the wedge, as a lesser perpendicular force DE ; the former being to the latter as AC is to HC . But this last perpendicular force on the side AC , is to that on the back which balances it, as AC is to AH , (by par. 11.) whence, compounding these ratios, the oblique force against one side of the wedge, is to the perpendicular

* If both these bodies be not hard, and their surface smooth, this parallel motion of the body along the plane will be prevented by the indenting, or by their roughness, that is, by the action of these bodies against each other. The original force in such a case cannot generate a separate motion in each body; both will then have one common motion just as if one body had pressed or struck the surface of the other perpendicularly.

lar force on the back which balances it, as AC^2 is to $AH \times HC$.

The oblique force fe , on the other side of the wedge, being equal to FE , will require another perpendicular force on the back to balance it, equal to the former perpendicular force; whence the whole force on both sides of the wedge, is to the whole force on the back, as AC^2 is to $AH \times HC$, or as the square of the side of the wedge, to the rectangle under half the back, and the perpendicular height.

23. There is no paralogism in this demonstration, and the conclusion would obtain in fact, if a machine could be constructed, in which all the circumstances should exactly agree to those here supposed, that is, to the *data* of the problem. If, for instance, two rollers could be made to press the sides of the wedge, in directions parallel to the back, but opposite to each other, and at the same time be permitted to roll down the sides of the wedge, with part of their original force; — but nothing like this is the case of the machines before mentioned, though intended to show experimentally the power of the wedge, in the circumstances laid down in this problem. Therefore the experiments made with those machines, though they undoubtedly succeed just as related, yet do by no means prove the proposition they are brought to prove, nor do they invalidate the reasoning here advanced. On the other hand, the problem, as it is here proposed, perhaps, will not suit any case in which the wedge can be *practically* introduced, but must ever remain a matter of *useless speculation*.

24. Nothing shows the difficulty of understanding clearly, and applying justly, the first principles of mechanics, more than the contradictory answers given to several plain problems, as well as to those relating to the wedge*.

But

* The reader will find two or three remarkable instances in the preface to Mr. Atwood's Treatise on Motion, and more might be given.

But the most singular instance is that of finding the force of the HUMAN HEART. Borelli makes it equal to *one hundred and eighty thousand pounds**. Dr. James Keil makes it *under eight ounces*†. And Dr. Jurin, (by virtue of *second fluxions*) finds it to be *just fifteen pounds four ounces, avoirdupoise*‡. Nor are the terms *Vis viva* and *Vis mortua* of the *Leibnitzian* school very intelligible. These indeed the disciples of *Newton* reject§; but they introduce others, such as *Vis inertia*, *Re-actio*, *Momentum*, &c., not always clearly explained. Well might the ingenious author of the *Analyst* tell these philosophers, that the term GRACE in Divinity, was full as intelligible as the term FORCE in their Philosophy; notwithstanding they so much boasted of *clear and unquestionable demonstration* §.

A P P E N D I X.

What follows is a description of two machines, intended to show the power of the wedge, in the case before mentioned. How far their circumstances answer to the *data* of the problem, in par. 20. is left to the reader's consideration.

25. Let a roller *F*, (fig. 11.) be drawn by a string *DF*, running over a pully *D*, so fixed, that when the roller bears against the side *BC*, of the isosceles wedge *ABC*, the string *DF* shall be parallel to *BC*. At the same time let this roller be drawn in the direction *FE* by another string *FE*, running parallel to *AC*, the back of the wedge. Let another roller *E* be applied in

* *Borelli de motu animalium. prop. 73.*

† *Keil's Tentamina Medico-Physica.*

‡ *Jurin's Essays, or Phil. Transf. Nr. 358.*

§ *Mac Laurin on Newton's Discoveries, B. II. Ch. 1. Sect. 12. and Ch. 2.*

¶ The controversy about the Force of bodies in motion, very much resembles that between the Jansenists and Molinists about Grace: both sides very hot, and very unintelligible.

in like manner to AB , the other side of the wedge. Lastly, let the force with which the rollers act against the sides, be balanced by another force P , acting perpendicularly to AC , the back of the wedge: then will the sum of the two forces on the sides, be to the force on the back, as AB^2 is to $AP \times PB$.

The rollers are here supposed to have no weight of their own, but to owe their whole force to weights that draw them.

26. The roller F is here acted upon by two forces, one of which being parallel to BC , opposes the motion of the roller along the side CB , but adds nothing to its pressure against the side of the wedge. The roller is also drawn in the direction FE , oblique to the side BC , but it can move only in the direction FP , perpendicular to the string FD , and therefore perpendicular to the side of the wedge.

27. Another machine, better adapted to practice, is described by Mr. *Ferguson* in his *Lectures*, page 67. Ed. 1760. Plate VI. Fig. 6. and is here delineated in Fig. 12. In this machine the wedge ABC , is not isosceles, but rectangular. The string FG (which must be always kept parallel to BC) may be either fixed to an hook, or made to run over a pulley H , in which case the cylinder F must be poised by a weight W . The equilibrium will be, when the weight of the cylinder F , is to the weight K , that draws the wedge along the horizontal plane (in the direction AI , perpendicular to the back AB) as BC^2 is to $AB \times AC$; as will be shown presently in par. 30.

28. The cylinder F is here acted upon by two forces, the weight W , and the force of gravity; the former only opposes the descent of the cylinder down BC , the face of the wedge, and creates no pressure of the cylinder against that face. The force of gravity acts in a direction parallel to the back BA , and therefore oblique to the face BC ; yet while the wedge is drawn along the horizontal plane, the cylinder F constantly rises in a line which is (relatively to fixed objects)

jects) perpendicular to the face BC , and can move in no other direction.

29. If the cylinder was suffered to descend freely down the face of the wedge BC , and the wedge was kept from running backwards (during the time of descent) by the weight K ; the case would then exactly answer the *data* of the problem, in par. 20. excepting that there would be here one oblique force only; the other two being perpendicular to the sides of the wedge on which they act.

30. From A let fall AD , perpendicular to BC , and from D let fall DE , perpendicular to AC . Let now BA represent the weight of the cylinder, then the force (of gravity) BA , may be resolved into the forces BD and DA . Again; the force DA may be resolved into the forces DE and EA . Now BA , BD , DE and EA , are to each other as BC^2 , $AB \times BC$, AC^2 , and $AB \times AC$ respectively: therefore if BC^2 represents the weight of the cylinder, then $AB \times BC$ will represent the force with which it descends down the face of the wedge; AC^2 will be the pressure of the wedge upon the horizontal plane, and $AB \times AC$, the force impelling the wedge (backwards) along the horizontal plane. — This last force will be greatest when AB and AC are equal, in which case it will be half the weight of the cylinder.

31. Let the string be fixed at H , and let the wedge be drawn under the cylinder; then, when the wedge has been drawn its whole length AC , on the horizontal plane, the cylinder F will be raised along the face of the wedge from C to D , whose perpendicular height, above the horizontal plane AC , is DE ; therefore the cotemporary velocities of the weights F and K , estimated in the directions in which they act, are reciprocally as AC to DE , that is, in a ratio compounded of AC to AD , and AD to DE ; or as BC^2 to $AB \times AC$, that is, reciprocally as those weights when they balance each other, according to the rule given in par. 20.

ESSAY

E S S A Y I I I.

ON SIR ISAAC NEWTON'S SECOND LAW OF MOTION.

THE design of Newton in the *Principia*, is to refer all that variety of motions we see in the universe to a few general principles. All philosophy consists in tracing out general principles by way of Analysis; and from general principles thus established, deriving synthetically a vast variety of particular phenomena *. Such general principles in *rational mechanics* †, are called rules or LAWS OF MOTION; they are circumstances which we see take place in the motion of all bodies we are conversant with, and are analogous to AXIOMS in abstract reasoning. Axioms are truths which the mind assents to, as soon as clearly understood; they cannot be *proved*, but are the basis of all demonstration. The enquiry into the relation which the various motions in the universe have to each other, not being an enquiry into the relation of abstract ideas; the first principles here, are not self-evident truths, but laws founded on universal experience. As in abstract science, so here; the fewer first principles are assumed by way of axioms, the more
scien-

* Optics, at the end.

† Rational or theoretical, as opposed to practical. See preface to the *Principia*.

scientific is the system of philosophy built upon them*. Again, the axioms of Euclid contain in them *virtually* all his geometrical propositions; for these propositions are only so many consequences derived from the axioms: so the laws of motion contain in them, *virtually*, a determination of the motion of all bodies in this our system, remote as well as near; for the particular circumstances of these motions may be derived from, and are necessary consequences of those general laws†.

But to shew that the motion of all bodies, in this our system, are consequences of the general laws laid down as axioms, we must compare these motions with respect to their quantities, their directions, &c. and shew that they have exactly (not in a loose way nearly) that very proportion to each other, which those laws demand. Now, whatever is thus made the subject of mathematical comparison‡, must have some mathematical measure of its quantity.

It

* It is not easy to determine what Newton meant by his *Æthereal Medium*, (Optics, Quere 19.) but it is certain he intended thereby to generalize further than he had done in his *Principia*. He thought he had discovered a grand universal principle, and that from ONE simple law he could derive, not only those kind of motions in bodies, we distinguish from others by imputing them to gravity; but also a vast variety of motions of another sort; distinguished, by being imputed to magnetism, electricity, &c. as their cause. See also preface to the *Principia*, near the end.

† And here we may observe the proper business of natural philosophy is not, in a strict sense, to assign the causes of motion, but to generalize and reduce all sorts of motion to as few classes as possible; thus to demonstrate the wisdom of HIM who imposed these laws upon matter, and has made all things in number, weight, and measure. See *Principia*, Def. VIII.

When we say that the motion of any bodies is caused by gravity, we mean only that their motion is such as, according to these laws, would be the consequence of a force, varying in a *reciprocal duplicate ratio of their distance from the central body*. If the motion of any bodies be of a different sort, we distinguish it by imputing their motion to some other unknown cause; such as what we call Magnetism, Electricity, &c. Thus effects which we do see and understand, are classed and distinguished, by the name we give to their supposed cause, of which we know nothing.

‡ Newton's calls his book, *The Mathematical Principles of Natural Philosophy*.

It must be observed here, that some quantities have a measure in themselves, and are therefore capable of mathematical comparison; while others, which admit of more or less, and that in all possible degrees, yet having no such measure, cannot be *mathematically* compared. Thus numbers have a measure in themselves, are made up of parts which bear a proportion to themselves, and to the whole; we can add one number to another equal number, and every one sees, the whole is just double of each part.

Extension has undoubtedly a measure in itself. The equality of two right lines is as intelligible, as the equality of two numbers. Euclid shows the possibility of adding one right line to another, or of subtracting one right line from another. 2 and 3 El. I. That the diameter of a circle is double the radius, is as intelligible, and as evident, as that twice two is four. In like manner surfaces are capable of addition, subtraction, and proportion, as appears from 41 and 47 El. I, and 1 and 2 El. II. So also are solids, as appears from El. XI.

Angles also, as well as lines, are capable of equality, of addition, subtraction, and proportion. An angle, as well as a right line, can be divided into two equal parts; 9 and 10 El. I. And the sum of the angles of every triangle is equal to two right angles; yet when we come to consider the proportion of one angle to another, we are referred to the parts of the circumference of a circle intercepted by those angles, as the measure of their ratio; 33 El. VI. Thus linear extension in a certain way, is the proper measure of the ratio of angles.

Equality of ratios is not questioned; El. V. Whether ratios can be added, subtracted, &c. in the same sense that lines can, is affirmed by some*; but doubted, or denied by others†.

Duration

* Namely, Saunderson, Halley, Cotes, and others See Saunderson's Algebra, Art. 293. Halley on Logarithms. Phil. Trans. Nr. 216. or the Abridgement, Vol. I. p. 108. also Cotes's *Logometria*.

† Namely, by Dr. R. Simson in his note on the 23. El. VI.

Duration seems to have no measure in itself. We are referred to the revolution of the earth round its axis, for a measure of time. Every intire revolution (marked out by the return of the same star to the same point in the heavens) is assumed as equal: and the parts of the time of every revolution, (as distinguished by the angular motion of the same star) are also supposed to be proportional to the angle described.

Whatever be the case of these quantities, there are others, undoubtedly capable of more or less, which yet have no measure of their magnitude in themselves; and are utterly incapable of that comparison, we call ratio or mathematical proportion. Neither do they admit of addition and subtraction, in the sense that lines and numbers do; nay their very equality is doubtful. Of this sort are all moral qualities of the mind, and all natural sensations of the body. Virtue and vice, are different in different persons; pleasure and pain, heat and cold, also admit of various degrees, but who can measure them*? Who can add one degree of heat to another equal degree of heat? Now to make quantities, which have not a measure in themselves, the subject of mathematical comparison, an arbitrary measure is assigned to them; by referring them to some quantity which has a measure, and to which they are related. Thus, he who graduates a thermometer, establishes an arbitrary measure of heat and cold. He refers the degrees of heat to extension, to which it has a relation, because heat increases the extension of most bodies; for he estimates the degrees of heat, by the *length* of that part of the tube, occupied by the quicksilver, as it is measured on his scale. And thus we may say, that the

* There are moralists, who not only compare *mathematically* the merit or demerit of one man with that of another, but also demerit with punishment. And in the Philosophical Transactions, N°. 293. a certain Doctor has *demonstrated algebraically*, that the doses of physics should be as the squares of the constitutions of the patients.

the heat of a moderate summer's day, when the thermometer stands at 64, is just double the heat at the beginning of a frost, when the thermometer stands at 32†.

In the same manner velocity or swiftness has no measure in itself. We perceive that some motions are swifter than others, but how shall we know when one motion is just twice or three times as swift as another? Now the space described in a given time depending on the degree of velocity, is with great propriety made the measure of velocity.

So also *Force* is not measurable in it own self. One force may undoubtedly be greater or less than another; and that whether its action is continued for some time, and is what we call pressure; or whether its action be instantaneous, or what is called impact. But who can determine when one spring pushes just as strong again as another? or one man strikes just as hard again as another? We can no more add one blow to another equal blow, than we can add the whiteness of one piece of paper to that of another just as white, and say it makes a colour as white again. Even the equality of many quantities, which have no measure in themselves; can hardly be ascertained.

Newton, perfectly aware of this, begins the *Principia* with establishing *measures* of all those quantities he intends to make the subject of mathematical comparison, and which have not a measure in themselves. Among these are quantity of matter, *Def. 1.* Quantity of motion, *Def. 2.* &c. all of which are referred for their measure, ultimately, to such quantities as have naturally a measure in themselves.

As motion is the consequence of a force acting on any body; so it is with great propriety made the measure of that force. Motion in itself indeed, has no natural measure; but an artificial one is assigned to it in the second definition.

By Newton's eighth definition, the quantity of the *vis centripeta motrix*, or such a force as acting continually,

† See this subject fully discussed in the Philosophical Transactions, N^o. 489.

nually, generates motion in any body, is to be estimated, (that is, its proportion to any other force of the like kind determined) by the quantity of motion generated by the action of that force, uniformly continued for a given time; or as it is usually expressed, the quantity of force is to be measured by the quantity of motion. As far as appears, the motion generated instantaneously, is also to be the measure of an instantaneous force. If therefore a body be already in motion, and that motion be increased or diminished by the action of some new force (of either kind) surely that adventitious force OUGHT to be measured in the same manner as the original force was. For would it not introduce the utmost confusion, to measure the original force by one rule, and the additional part by another? And if we lay it down as a *definition*, that the change of motion is the proper measure of the force so impressed, is it not identical to lay it down also as a *proposition* (in the second law of motion) that the change of motion is proportional to the moving force impressed? And what is the proof? namely, "that if any force generate a particular degree of motion, a double force will generate a double motion; whether in the way of pressure or of impact." This surely, if any thing can be, is identical. For what is the *definition* of a double force? — That which generates a double motion*.

What has been said may be applied to the other part of the second law of motion; "that the change of motion is produced in the direction of the right line in which this force is impressed." For there is no other mark to know in what direction any force is impressed, but the direction of the motion produced in consequence of that force. As the quantity of motion

* The common aphorism, that "causes are proportional to their effects" is inapplicable, when we have no measure of either; and identical when one is measured by the other. I speak of proportion in a mathematical, not moral sense. See Mac Laurin's Account. B. II. Ch. 2, Sect. 22.

motion produced, is made the *measure* of the *quantity* of force; so the direction of the motion is ever the *mark* of the *quality*, or direction of the force.

Newton certainly intended this law of motion should be a more general proposition, than what he derives from it in his first corollary (*accedet igitur corpus, &c.*) Whatever that proposition was, he seems to have thought that it followed *a priori* from the nature of motion. For he does not establish this law, as he does the other two by appealing to facts; but by arguments drawn from the nature of motion. Whether this attempt to go higher, and generalize more, has not introduced obscurity into the law itself, and embarrassed the argument by which he endeavours to prove the truth of the first corollary, (all that is material to be established) let others judge. What follows is an attempt to express this law, in more intelligible (though perhaps less general) terms, and to put its proof, like that of the other laws, upon universal experience.

DEFINITION.

By the *determination* of a force, is meant either the line of its direction, or any other line parallel to it*.

LAW.

When a body is acted upon at once by two forces, its quantity of motion estimated in the respective determinations of those forces; is the same when they act jointly, as when they act separately.

To explain this, describe the parallelogram $ABCD$, (fig. 14.) and draw the lines EE , FF , parallel to AC ; also GG , HH , parallel to AB .

Let a body A , be acted upon in the direction AB , by a force that would carry it in a given time, through the line AB . Again, let the same body be acted upon in the direction AC , by another force, that would carry

* This definition is borrowed from Saunderson's Introduction to Fluxions.

carry it in the given time through the line AC . Let now these two forces act at the same time jointly on the body; then it will be carried just as far by their joint action in the determinations GG, HH, CD , as it would be by the former force alone, in the same determination, or in the line AB . It will also be carried just as far in the determinations EE, FF, BD , as it would by the latter force alone in its determination, or in the line AC . The action of the former force, though mixed with the action of the latter, will yet have no effect upon the body, but in its proper determination GG, HH, CD ; nor will its effect, *thus estimated*, be increased, or diminished, or the body be made to approach to, or recede from, the line BD , by the accession of the latter force. In like manner neither will the action of the latter force (so estimated) be changed by being joined with the former. Its effect upon the body in its own proper determination EE, FF, BD , will not be altered; nor will the body, by the joint action of the former force, be made to approach to, or recede from, the line CD , parallel to AB . Thus we may say, neither the quantity nor the quality of the motion generated, will be at all altered by the joint action of two forces upon any body: it will be the same (so estimated) as if they had acted separately.

What has been said is only to illustrate the meaning, not to prove the truth of this law.

SCHOLIUM.

The rule which those motions follow, that are occasioned by the action of a single force, is laid down in the first law of motion: in this second law, we have the rule which motions arising from the combined action of two forces (of either kind) always follow*. Newton explains (in his first corollary) those

* The third law respects the communication of motion from one body to another; whether instantaneous, or gradual.

those circumstances which always attend these motions, as we have done, (*nihil mutabit velocitatem*, &c.); but then surely these circumstances should have been the law itself, not a consequence from it, implied in the words *Nam quoniam vis*, &c. — This law respects the motion of a body so acted upon, as it is estimated in the determination of each force separately; but from thence, the line which the body will describe, and its motion estimated in its own proper path, may be derived as follows.

COR. 1. At the end of the time in which the body by the former force would have described the side *AB*, or by the latter force the side *AC*; the body when acted upon by the two forces conjointly, will be found in *D*, the opposite angle of the parallelogram. For by this law the former force does not change the approach of the body to the line *CD*, therefore at the end of the time it will be found somewhere in that line. By the same law, the latter force does in no wise change the approach of the body to the line *BD*, therefore at the end of the time it will be found somewhere in that line also; and therefore be found in their concurrence *D*, the opposite angle of the parallelogram.

S C H O L I U M.

As this law obtains in the joint action of two forces, of either kinds (instantaneous or continual) the body will in either case be found in the point *D*; but the road by which it gets to *D*, and its rate of travelling, will be different according to the nature of the forces combined together.

COR. 2. Draw *AD*, the diagonal of the parallelogram, and let the two forces act instantaneously. I say in this case the body *A*, will describe the diagonal *AD*, with an uniform motion, and in the same time in which it would have described the sides by these respective forces separately. For as the action of

such forces ceases after the first impressi^on given; whatever direction and velocity these conjoint forces give the body, it will go on uniformly with that velocity, and in that direction ever after, by the first law of motion. Now at the beginning of the time, the body was supposed at *A*, at the end of the time it will be found in *D*, by Cor. 1. therefore it will have described a right line passing through *A* and *D*, (that is the diagonal *AD*), with an uniform motion, and in the same time that by the former force or impact, it would have described the side *AB* uniformly, or by the latter impact the side *AC*.

COR. 3. Let both forces act continually, but uniformly, during the time in which the body by one force describes *AB*, by the other force *AC*. In this case, when the sides are described separately, they will each be described with a velocity uniformly accelerated; and because the time in which these spaces are described, is given, the accelerating forces will be as those spaces, or the sides *AB* and *AC*, by the first law of motion*. I say in this case, the body when acted upon by both forces conjointly, will describe the diagonal *AD*, with a velocity uniformly accelerated, and in the same time in which it would have described either side by its respective force separately.

In the lines *AB* and *AC*, let *E* and *G* be two cotemporary places of the body, when it describes those sides separately. Compleat the parallelogram *AEPG*, and *P* will be the cotemporary place of the body, when the forces act conjointly, by Cor. 1. But because *B* and *C* are also cotemporary places of the body, when the forces act separately, we have by the first law of motion as before $AE : AG :: AB : AC$. The parallelograms *AEPG*, and *ABDC*, are therefore similar†, and the point *P* lies in the diagonal *AD*.

* See Saunderson on the descent of heavy bodies, Cor. iv, Canon first.

† Def. 1. El. VI.

AD. In like manner, the place of the body at every instant, while the forces act conjointly, will be found in the diagonal. Moreover, $AB:AD::AE:AP$. If therefore the point *E* (representing the place of the body) describes the line *AB*, with a velocity uniformly accelerated, the point *P* will also describe the diagonal *AD*, with the like kind of motion, and in the same time. Therefore the body, when acted upon by both forces jointly, will describe the diagonal with a velocity uniformly accelerated, and in the same time in which it would have described either side by its respective force separately.

S C H O L I U M.

The motion of the body, along the diagonal generated by the conjoint forces, is just such as would have been produced by a single force, acting continually and uniformly in the direction of the diagonal *AD*, and whose quantity is to either of the other forces, as the diagonal is to the respective side. For the time being given, these accelerating forces, are as the spaces described, as was said before.

COR. 4. Hence, a third force acting in the contrary direction *DA*, and whose quantity is to either of the other two forces as the diagonal, to the sides respectively, would just counter-act the joint effect of the other two forces, and keep the body in equilibrio.

COR. 5. Hence, if a body be acted upon by three forces, whose quantities and determinations are represented by the sides of a triangle *ABD*, that body will be at rest.

S C H O L I U M.

If the forces are one instantaneous, and the other continual and uniform, the body at the end of the given time will be found in the opposite angle of the parallelogram, as before, but the path it will have described

described will not be a right line, but a parabola, as is shewn by all the writers on projectiles.

It may now be asked, how can such a law as this be proved experimentally? By what sort of trials are we to find out, whether the determination of two forces, or the quantity of motion produced in such determination, is, or is not, altered by their acting in conjunction? It may be answered, that all the immediate consequences of this law, that can be established by experiment, are so many proofs of the principle from which they are derived. Nothing is more easy than to apply three weights so as to act at once upon one body, and that in any proposed directions. Where-ever the quantity and directions of these forces, are so proportioned, as according to the corollary drawn from this law, to balance each other; there we shall find the body will be at rest. All the experiments made on what are called the mechanical powers, are also (by consequence) proofs of this law.

The first law of motion cannot be proved by direct experiment. We never in fact do see any body whose motion always continues uniform and in a right line; but we justly infer from such experiments as we can make, that it would do so were all impediments removed. Paradoxical as it may seem, yet it must be acknowledged, that those circumstances which it is alledged do take place in all motions, and therefore become *laws of motion*, are yet not to be seen (simply) in any one motion we are acquainted with. By making these laws so very general and abstract, they exist in idea only; they do not admit of a direct proof from simple fact, and can only be inferred by way of consequence from facts. An experiment cannot be contrived in which a body shall be placed in those very simple circumstances, supposed in the law itself; foreign ones will attend it. But though some argument must be used, it is seldom that more than one step is necessary, all beyond is matter of fact; so that these laws may be fairly said, not to depend on metaphysical

physical reasonings about matter and motion, or uncertain hypotheses; but as all sound philosophy should do, on observation and experiment.

Mac Laurin in his comment on this law, seems to represent it as affirming, that when a force acts in part upon a body *A*, it produces only so much motion as the acting part, not the whole, can produce. There are many instances where a force, which seemingly acts upon one body, really acts upon two. When the motion, produced in the second *B*, is so slow as to be imperceptible, we then say part of the original force is sustained by *B*, and part only acts upon *A*; but surely to say the motion generated in *A*, is proportional to so much force as is impressed on *A*, or that "the acceleration is proportional to the power that acts," is identical. For we undoubtedly measure, the power that acts, by the acceleration it produces. It is identical considered as a law or rule. — It is quite another thing to enquire how we shall determine what proportion that part of the force which does act upon *A*, bears to the whole.

When *Mac Laurin* explains Newton's first corollary he *premises* this principle. "That when a body "is acted upon by a motion or power parallel to a "right line given in position, this power or motion "has no effect to cause the body to approach to- "wards that right line or recede from it, but to "move in a line parallel to that right line only, as "appears from the *second law of motion*." It is not easy to understand this, if the body be supposed to be acted upon by one force only: for then the first law is the rule of its motion. And if it relates to the case of a body, acted upon by two forces at once, (the case considered in the second law), it will, when fully explained, probably coincide with the principle or law before laid down. That this *principle* is or should be the law itself, we contend; but if this be denied, then we ask, is this *principle* a lemma to be *premised* for demonstrating the law, as is said at first; or a *consequence*

sequence from the law, as is said at the conclusion? See Mac Laurin's Account, Book II, Ch. 2. Sect. 3. & 8. p. 115. and 121. Quarto*.

What Mr. *Milner* says in his excellent and judicious comment on the third law of motion, may be fairly applied both to Mac Laurin, and even Newton himself, with respect to the second law of motion. —

“ I cannot possibly conceive that so skilful and accurate a philosopher could believe, that the *third* law of motion was an inference of reason, exclusive of all experiment; and yet, if words have any meaning at all, the above quotation inclines us to think so. A law of nature is not merely a deduction of reason: it must be proved, either at once and directly, by some simple and decisive experiments; or if that cannot be done, by such experiments as enable us to collect its existence by the assistance of geometry.” See Phil. Transf. Vol. LXVIII. Part I, page 352.

What has been said, is by no means intended to cavil at so great a man as Newton; but inaccuracies in the manner of reasoning create great difficulties to learners, and sometimes to more experienced philosophers. Of this we have a remarkable instance in the controversy, carried to a great length, between two able mathematicians, ROBINS, and JURIN, about Newton's *Proof* of his first Lemma to the laws of motion, owing to this; that Newton took that for a proposition, and therefore to be *proved*, which is certainly nothing but a definition, and therefore can only be *explained*.

* Sect. 11. and 12. p. 123. of Mac Laurin's account, are rightly termed illustrations, not proofs, of the second law of motion. They *suppose* this principle or law, to be true.

ESSAY IV.

PROPERTIES OF THE CYCLOID.

LEMMA.

LET TP (fig. 15.) be a tangent to the circle DPE in P ; and DE a diameter. From P let down PM , perpendicular to DE ; draw the chord EP , to the extremity of the diameter, and it will bisect the angle MPT , made by the perpendicular PM , and the tangent PT .

Draw the chord DP , to the other extremity of the diameter, and $TPE = PDE$ (32. El. III.) $= MPE$ (8. El. VI.). Therefore the chord EP , bisects the angle MPT .

DEFINITIONS.

If a circle DPE be made to roll (suppose from the left hand to the right) against a straight line ALB , until the point P , which at first touched the line in A , comes to touch it again in B , then the line $AFP B$, described by the point P , is called a CYCLOID; the circle, *the generating circle*; the line AB , *the base*; FL perpendicular to the middle of the base, *the axis*; and (supposing the axis to be perpendicular to the horizon) F is *the lowest point* of the cycloid.

Let C be the center of the generating circle, D the point

point of contact with the base, in any supposed position of that circle. Draw DE a diameter passing through the point of contact, and therefore perpendicular to the base AB . Draw TPt a tangent to the circle in the describing point P . Draw MPm through P , perpendicular to the diameter DE , and parallel to the base. Lastly, from P , draw the chords PD and PE to the extremities of the diameter DE , draw also the radius CP .

SCHOLIUM.

In the generation of the cycloid, the motion of the describing point is compounded of a circular and rectilinear motion. For while the circle turns round the center C , that center, (and every other point in the plane of the circle) is carried forward in a right line, parallel to the base AB . By the former motion alone, the point P , would be carried in the direction of TPt , the tangent to the generating circle in P : by the latter alone, the point P , would be carried along the right line MPm , parallel to the base; by both motions conjointly; the point P will be carried in an intermediate direction; that of EPp , a tangent to the cycloid in P .

PROP. I. Let the little line Pt represent the circular velocity, and Pm the rectilinear velocity of the describing point P . Now these are equal; therefore take Pt and Pm equal, complete the parallelogram $Ptpm$, draw the diagonal Pp , and it shall represent both the direction and velocity of the describing point, when carried on by both motions jointly. This follows from the known laws of motion*.

COR. The angle $tPp = mPp$; therefore the diagonal continued, bisects the angle MPT , and coincides with the chord PE , by the lemma.

Also the other chord PD is perpendicular to EPp , the tangent to the cycloid in P . (31. El. III.)

SCHOLIUM.

- * Newton's Princip. Cor. Laws of Motion.

SCHOLIUM.

That the chord DP is perpendicular to the tangent, is sometimes proved from this consideration; that the describing point P , may be looked upon, as turning round the point of contact D , as a center; and from hence, some have concluded *falsely*, that D is the center of curvature, and DP the radius of curvature, of the cycloid, at the point P . The solution of the following problem, will set this matter in a true light.

PROBLEM.

To find the center of rotation, of the chord line DP , always drawn through the point of contact of the generating circle, and the describing point P .

Conceive the point of contact transferred (by the rotation of the generating circle) from D to e . And here let it be observed once for all, — that the rectilinear or progressive motion, *First*, of the point of contact; *Secondly*, of the center C , of the generating circle; *Thirdly*, of the other extremity E , of the diameter DE ; and *Fourthly*, of the describing point, are all four equal, and in right lines parallel to the base. Take therefore $De = Pm$, and while one end D , of the line DP , is transferred from D to e ; the other end P , will be transferred from P to p , and a line passing through p and e , will give the position of a new chord line very near the former. Resolve the motion along De into Dd , perpendicular, and de parallel to Dp ; and Dd is the perpendicular translation of the end D , while Pp is the perpendicular translation of the other end P , of the chord line DP .

Draw the other diagonal mt , and the two diagonals will bisect each other, perpendicularly in n . Now the triangles mPn and eDd are equal, and $Pn = Dd$, or $Pp = \text{twice } Dd$; therefore the new chord line drawn through p , e , and d , will meet the old one PD , produced in a point whose distance from P is equal to
twice

twice PD . And this is the real center, about which the chord line PD revolves; and is the true center of curvature of the cycloid at the point P , the radius of curvature being equal to twice PD .

REMARK. If the point D be considered as a point always fixed in the circumference of the generating circle, then it is undoubtedly stationary at D ; the circular and rectilinear motions of that point being then equal, and in opposite directions; but if D represents the point of contact (where-ever it may be) then D is not stationary.

COR. 1. The radius of curvature at F , is twice FL , or twice the axis of the cycloid.

COR. 2. All things as before; the line nt represents the velocity with which the chord PD shortens; the line Pn , the velocity with which the chord EP lengthens; and Pp , the velocity with which the cycloidal arch FP increases. Now Pp is always double of Pn , and at F , both the cycloidal arch FP , and the chord EP are nothing; that is, they begin together, and the arch increases every where, with double the cotemporary velocity of the chord. Therefore the cycloidal arch FP , reckoned from F , is double the chord EP .

REMARK. It may seem strange, that Pp should represent the velocity with which the arch FP increases; yet not represent the velocity with which the chord EP increases. For the describing point P is the common termination of the arch FP , and of the chord EP ; and the point p is the common termination of the new arch and new chord, and therefore Pp their common increment.

In this representation of the case, both the chord and the arch are referred to absolute space; but the chord ought always to be referred to the rolling circle. Both the arch Fp and chord Ep , will have one common termination in absolute space, but while P is carried by a motion, whose velocity and direction is Pp , the other end of the chord E will also be carried

ried (in absolute space) by a motion whose velocity and direction is represented by De or Pm . Let this (as before) be resolved into two others, Pn and mn , and Pn will represent the velocity of the point E , in the direction EP , and Pp will represent the velocity of the point P , in the same direction, *considered absolutely*. But the increase of the chord line EP , depends on the *relative* velocity of its two boundaries E and P . If they had one common velocity in the direction of the line EP , that line would receive no increase of its length. To find the velocity with which the line EP increases, we must subtract the absolute velocity of the point E , from the absolute velocity of the point P , in their common direction; that is, we must take Pn from Pp , and the remainder $np = Pn$ is the velocity with which the chord EP increases, and is half Pp , the velocity with which the arch increases; because F , the other end of that arch, is a fixed point.

If (as we ought) we refer the point p to the generating circle; t will be the place where it will be found in that circle. The diameter DE always passes through the point of contact; therefore relatively to this moving circle, Et and Dt , are the new chords, when the describing point is at p in the cycloid, that is at t in the circle; and Pn is the increment of one chord, and nt the decrement of the other, as before.

PROP. 2. The accelerating force of a body oscillating in a cycloid, is as its distance from the lowest point F .

For the accelerating force in any point P , is the same as if the body was placed on the tangent to that point EPp , or on the chord EP . But the accelerating force of a body, placed on different chords of a circle, all drawn to its lowest point, is as the length of those chords by the laws of motion*.

Therefore

* The accelerating force, is to the force of gravity, as EM to EP ; or as EP to DE ; but DE is constant; therefore the accelerating force is as the chord EP . See Cotes *De Descensu Gravium*, or Keil's *Physica*.

Therefore the accelerating force at any point P of the cycloid, is as the chord EP , or as double that chord, or as the length of the cycloidal arch FP , by cor. 2. to the problem.

COR. Hence the times of all oscillations in a cycloid, whether through greater or less arches are equal, by the laws of motion*.

PROP. 3. If the center of the generating circle be carried uniformly forward, with half the velocity, that is acquired by an heavy body in falling from rest, through its diameter; the describing point will every where have the velocity of a body oscillating in the cycloid.

Because DPp , and CPt , are each right angles, take away the common angle DPt and $DPC = tPp$, and the isosceles triangles CPD and tPp , are equiangular, and $Pt (= Pm)$ is to Pp as CP to DP . Therefore if CP represent the rectilinear velocity, with which the center C is carried forwards, DP will represent the velocity wherewith the point P describes the cycloid. $DP^2 = DE \times DM$ (8. El. VI.) and DP is as $\sqrt{DM}$, therefore DP is always proportional to the velocity acquired by an heavy body in falling (from rest) through the space DM †. At the point F , the line DP becomes the line DE , and the space DM , becomes the space DE . Therefore as DE is to DP ; so is the velocity acquired in falling through DE , to the velocity acquired in falling through DM ; and as $\frac{1}{2}DE$ or CP , is to DP , so is half the velocity acquired by falling through DE , to the velocity acquired by falling through DM ; but it was before shown that CP is to DP , as the rectilinear velocity of the center C , to the velocity wherewith the point P describes the cycloid. Hence, the rectilinear velocity of the center, is to the velocity of the describing point, as half the velocity acquired by falling through DE ,

* *Princip. L. I. P. xxxviii. Cor. 2. or Keil's Physics, Th. XLV.*

† *Cotes De Def. Grav. Th. II. or Keil, Th. XVII. Cor. 6.*

DE , is to the velocity acquired by falling through DM . Now if the antecedents be made equal, the consequents will be so too; that is, if the velocity of the center be made equal to half the velocity acquired by falling from rest through DE ; then the velocity of the describing point, will be equal to that acquired by falling from rest through DM , and this (by the laws of motion*) is the velocity acquired by a body, descending from rest, through the cycloidal arch BP . Thus, if the center of the generating circle, be carried uniformly forward in a right line, parallel to the base, with the velocity aforesaid; the describing point P , will have the same velocity every where, that a body oscillating in the cycloidal arch has.

COR. 1. In this case, the describing point will trace out the whole cycloid, exactly in the *same time*, that a body takes in oscillating through it. For their velocities at the same points are precisely the same.

COR. 2. In this time (of the description of the cycloid) the center of the generating circle, by its uniform velocity, goes through a line equal to the base or circumference of that circle: and (by the laws of motion†) with this same uniform velocity (half that acquired in falling through DE) a body would be carried through DE , in the time of descent. But the former time, is equal to the time of one oscillation through the cycloid (by Cor. 1.) and the latter time is the same, as the time of descent through the axis of the cycloid (for $DE=LF$); therefore the time of one oscillation, is to the time of descent through the axis; as the circumference of the generating circle, is to its diameter.

COR. 3. Hence, the time of one oscillation in different cycloids, is in a subduplicate ratio of their axes‡.

SCHOLIUM.

* Cotes, *De Desc. Gr.* Th. V. or Keil's *Physics.* Th. XXXVIII.

† Cotes, *De Desc. Gr.* Th. II. or Keil, Th. xvii.

‡ Cotes, *De Desc. Gr.* Th. II. or Keil, Th. xvii. Cor. 6.

SCHOLIUM.

We before observed, that the center of the circle, having the same curvature with the cycloid at the lowest point *F*, was in the axis produced, and that its radius was twice that axis. Therefore a body oscillating in such a circle through a very small arch, (that is a pendulum whose length is double the axis) may be considered as oscillating in that cycloid, and what has been said of one, may be applied to the other.

ESSAY V.

ON DEFINITION I. AND COR. I. PROP.
X. WITH COR. I. PROP. XIII. BOOK I.
OF NEWTON'S PRINCIPIA.

ON DEFINITION I. B. I. OF NEWTON'S PRINCIPIA,

A DIFFICULTY occurs at the very first entrance on this curious and wonderful book. The quantity of matter in any body (says Newton) is to be measured by its density and magnitude together. The magnitude or solid content, being a mode of extension, has clearly a measure in itself: but what is the measure of density? Will you say that density is as the weight directly and magnitude inversely*? Why then not say at once, that the quantity of matter in any body, is to be measured by its weight alone; without any regard either to density or magnitude? And it might seem so by those words, *innotescit ea per corporis cujusque pondus*. But it is not Newton's intention, by virtue of a definition, to constitute *weight* the measure of *quantity of matter*. He gives a *reason* why the quantity of matter in every body is known

* Mac Laurin's Account of Newton, p. 107.

known by its weight. *Nam ponderi proportionalem esse reperi per experimenta.* This then is a proposition to be proved; although the definition of the term, *quantity of matter*, is not yet settled, and nothing has been said about *forces*.

Let us consider the experiment referred to by Newton; and see what can be deduced from it. Let then two bodies descend from rest, for a given time; at the end of this time they will have acquired equal velocities. This is all you see in the experiment; now for the argument. If then you *please* to measure quantity of motion, by velocity and quantity of matter conjointly, according to Definition II; you may *say*, their quantity of motion, is as their unknown quantity of matter: and if you further *please* to measure quantity of force, by the motion it generates in a given time, then you may also *say*, that their unknown force of gravitation, is as their unknown quantity of matter. That is, as the quantity of matter in one body is to the quantity of matter in the other; so is the force of gravitation in the former body, to the force of gravitation in the latter. The antecedents of the terms of this proportion, are to be fixed by experiment. Thus — There is a certain unknown quantity of matter in a lump of brass, whose natural force of gravitation is *called* one pound. Experiment shows that this force, exerted on this quantity of matter, for one second of time, generates in it, a velocity which (uniformly continued) would carry it through 32 feet in that time*. And because this very same velocity (of 32 feet in a second) is generated in the same time, in all other bodies, by their respective forces of gravitation; it is concluded, as before, that the proportion of the quantity of matter in this lump of brass, to the quantity of matter in *any* other body; and also the proportion of the force of gravitation in this lump of brass, to the
force

* It is in strictness $32\frac{1}{2}$ feet, but we take round numbers. See this afterwards proved.

force of gravitation in that other body, is the same. But this proportion though the same, is at present unknown; and we are yet to seek either for a measure of the quantity of matter in any two bodies, or for a measure of their force of gravitation. Thus much we have obtained, we need not seek both: but to measure the unknown quantity of matter, by quantity of force, and the equally unknown quantity of force, by quantity of matter, is to argue in a circle.

Now though the proportion of the force of gravity in any two bodies is yet unknown, yet the equality of that force in a particular case may be experimentally ascertained thus. Let there be two bodies, one of lead, and one of wood, of such a shape that the proportion of their solid contents can be experimentally known by mensuration. Let the leaden body be exactly counterpoised in a pair of scales. Remove the leaden body, and put the wooden one into its place. Alter its size (keeping the form regular) 'till it exactly balances the same counterpoise in the opposite scale, that the leaden one did. Now when we see in fact, that the effect of their weight is the same, we must admit, that the force of gravitation in each body is equal*.

Now, if the force of gravitation be equal, it follows from the experiment on falling bodies, that (according to Def. II. and VIII.) we must account the quantity of matter in these two bodies equal. Call this 1. Take D to d , in the inverse ratio of their magnitude, when their weights are equal; that is, let D be to d , as the magnitude of the wooden body, to the magnitude of the leaden one; but call D the density of the leaden body, and d the density of the wooden one. Suppose now the wooden body reduced to

* The balance is (in this way) a test of the equality of two weights: but at present it wants a proof, that the balance is a test also of the proportion of two weights; that is, of the force of gravitation of two bodies. Another measure of that force, (and forces in general) is laid down in def. VIII.

to the same size with the leaden one; then will its quantity of matter be reduced in the ratio of D to d ; that is, as D is to d , so is 1 to $\frac{d}{D}$, and $\frac{d}{D}$ will be the quantity of matter in this fictitious wooden body, whose magnitude is equal to the magnitude of the leaden one: and this, because in two homogeneous bodies, (such as the two wooden ones) the quantity of matter is proportional to their magnitude*. We have then two bodies equal in magnitude, and have shown (by consequence) from the experiment on falling bodies, that the quantity of matter in the leaden body, is to the quantity of matter in the wooden body, as 1 to $\frac{d}{D}$, or as D to d . Now, if the magnitude of the leaden body be changed, its quantity of matter will also be changed proportionally; because in homogeneous bodies, quantity of matter is as their magnitude. The same is true also of the wooden body. Let then M be the magnitude of the leaden body, and let m be that of the wooden one, and their quantity of matter will be as $M \times D$ to $m \times d$.

We have then a definition of density. The inverse ratio of the magnitude of two bodies, having experimentally equal weights; constitutes what is to be called the ratio of their densities.

It will be said, that we introduce definitions II. and VIII. to explain definition I. It is most likely, the measure of force was settled first; and quantity of motion was assigned for its measure. Quantity of motion must of course be settled next: velocity and quantity of matter are appointed for its measure. Magnitude and density, are constituted the measure of quantity of matter. Magnitude, being a mode of
extension

* This is the definition of homogeneous bodies.

In considering the quantity of matter in any body, we must admit the possibility of there being more or less in the same space; that some bodies are homogeneous, others heterogeneous, and other circumstances of density.

extension, has a measure in itself, and density is determined by experiment as before. For we must ultimately build upon experiment, or the whole fabric would be visionary, and have no relation to nature. In teaching philosophy scientifically, these definitions must be ranged in the contrary order; beginning with those quantities, whose measure is in nature; using in the subsequent definitions, no term but what has been before explained.

We have before observed, that in every system of gravitation, this rule obtains, viz. as the quantity of matter in one body, is to the quantity of matter in another; so is the force of gravitation in the former body, to the force of gravitation in the latter *. And also that the antecedents of this proportion, settled experimentally, constitutes every particular system of gravitation. We come now to consider the comparison of different systems of gravitation.

To explain this, suppose the lump of brass, formerly mentioned, and weighing one pound at the surface of the earth, to be carried up to such a distance from the earth, as to weigh only half a pound †. This force (half the former) exerted on the same lump of brass, for one second of time (as before) would generate in it only half the former velocity (Def. VIII.) that is, a velocity which uniformly continued, would carry it through, (only) 16 feet in a second of time. But if the force of gravity in these upper regions, is also *proportional to the quantity of matter*, then all other bodies, of what sort soever, will by the force of their respective gravities in those upper regions, also acquire the same velocity in one second of time, viz. a velocity of 16 feet in a second; just as all bodies whatever, at the surface of the earth, acquire a velocity of 32 feet in a second.

There-

* It is this very circumstance that constitutes what is called a *System of Gravitation*.

† We suppose the force with which the earth attracts all bodies to be lessened, as their distance from it is increased.

Therefore, to compare one system of gravitation with another, we need not make our experiments on the same lump of brass, or same body, or even regard quantity of matter: for the velocity generated in the same time in any two bodies whatever (one in each system) will be the same respectively as would be generated in two other bodies (one in each system) but having equal quantities of matter. And in that case it is evident, that the general force of gravitation, is as the velocities generated in these equal bodies, in equal times, by def. VIII.

Hence it is, that in all Newton's propositions about centripetal forces, no regard is had to the quantity of matter in the revolving bodies. The very same velocities, that is, the very same linear motions will be generated by those central forces, whether the attracted bodies be great or small, and whatever be the density of the matter of which they are composed. Their motions will depend on what is called their *vis acceleratrix* only.

But if it be true, that the particular gravitation of every body in every system, depends not only on the quantity of matter in the body so attracted, but also on the quantity of matter in the attracting or central body (it being *cæteris paribus* proportional thereto) then the *vis acceleratrix* of the attracted body, is as the quantity of matter in the central body; and on this principle is founded the enquiry into the quantity of matter, density, &c. of the sun, and such of the planets, as have satellites revolving round them. Book III. prop. 8.

Let us now consider in what way the velocity, acquired by any body in falling from rest is derived from the motion of pendulums: for it is impossible to find this by direct experiment.

Let then the body in question, be suspended by a string, or rod, and made to oscillate as a pendulum; and call the length of the pendulum
rod

rod L^* . Let the periphery of the circle be to its diameter as p to 1. I say then, that $L \times pp$ will be the velocity which that same body will acquire in falling from rest for the time of one oscillation. Or thus. p is equal to 3,1416; therefore $pp = 9,8695$. Now, 9,8695 times the length of the pendulum, will be the line that would be described with the velocity acquired (by the oscillating body) in the manner aforesaid; provided that velocity was uniformly continued for the time of one oscillation.

For p is to 1, as the time of one oscillation, or t , is to the time of descent through $\frac{1}{2}L$ †; which time therefore is $\frac{t}{p}$. Again, it will be as the square of the

time of descent through $\frac{1}{2}L$, or $\frac{tt}{pp}$; is to the square of the time of one oscillation, or tt : so is the space descended through in the former time, or $\frac{1}{2}L$; to the space

* This length must be reckoned from the center of suspension to the center of oscillation.

When the pendulum is applied to a clock, the clock-weight acting through the wheels, preserves the motion of the pendulum. But this is the least valuable property of the clock. The wheels serve also to count the number of oscillations in a siderial day, and hence is derived the time of one oscillation. Thus, a siderial day is 23 hours and 56 minutes, or 86160 seconds. Therefore if the pendulum makes 86160 oscillations in a siderial day, it swings *seconds*.

It is probable, the first inventor of pendulum clocks meant no more than to contrive a machine to count the oscillations of the pendulum; and it became accidentally a TIME-KEEPER. *Hevelius* and others, complain of the trouble of counting the oscillations of the pendulum, while they used it in its detached state in their astronomical observations. See *Hevelius's Machina Cælestis*; and the experiments of the *Academie del Cimento*, by Waller, p. 10.

† See the latter end of Essay IV. The rules respecting the descent of heavy bodies acted upon by an uniform gravity, and the rules respecting the motion of pendulums, are here supposed to be known. These rules are clearly demonstrated by Mr. *Huygens* in his *Horologium Oscillatorium*, which in the opinion of a very able judge, Mr. *Baron Maseres*, is the model of good writing on a subject of mathematical philosophy.

space descended through in the latter time, or the time t ; which space therefore is $\frac{1}{2}L \times pp$; but in this time of descent (or the time of one oscillation) the body by its last velocity, uniformly continued, would be carried through double that space, or $L \times pp$, which was to be shown.

Let L be the length of a pendulum, having a leaden ball, and oscillating seconds at Paris; and let l be the length of a pendulum, having the same leaden ball, oscillating seconds under the equator. Then is $L \times pp$ the velocity which this ball would acquire in one second, in falling from rest at Paris; and $l \times pp$, is the velocity acquired by the same ball in falling in like manner at the equator. Let q be the quantity of matter in this leaden ball, which whatever it be, is the same both at Paris and at the equator. Then by definition II. $L \times pp \times q$, is the quantity of motion generated in one second by the force of gravitation at Paris; and $l \times pp \times q$ is the quantity of motion generated in the same time by the force of gravitation at the equator. Therefore by definition VIII. as $L \times pp \times q$ is to $l \times pp \times q$, so is the force of gravitation at Paris to the force of gravitation at the equator; which is therefore as L is to l , or as $36 \frac{8,555}{12}$ is to

$36 \frac{7,468}{12}$, or as 2295665 to 2290000. See Principia, L. III. prop. 8.

There is no necessity in this case, to suppose the experiment made on the same pendulum. Suppose the pendulum ball at Paris had been of gold, and of any size: the time of one oscillation of this golden ball at Paris, will be the same as that of the aforesaid leaden ball at Paris. Quantity of matter makes no difference in the time of one oscillation, nor in the velocity generated by the force of gravity, when that force is *proportional to the quantity of matter*, as has been observed. And this rule obtains both in the system

system of gravitation of all bodies at Paris; and in the gravitation of all bodies under the equator.

To return to our subject. The other definitions have more or less somewhat of an obscurity in them. The sixth in particular is liable to the same objection with the first. For what is *the efficacy of a cause*? This definition is not quoted any where, nor does there seem to be any necessity for establishing a measure of the absolute quantity of centripetal force; but if there be, the *efficacy of the cause that propagates it*, cannot be applied to this purpose, as wanting a measure itself. *Mac Laurin**, and after him Mr. *Thorpe* says, "The absolute quantity of centripetal force, is measured by the motion generated in the same time, in equal bodies placed at equal distances from different central bodies."†. But neither can this be admitted, unless the LAW of the variation of those centripetal forces be the same. If one centripetal force be in an inverse duplicate, and the other in an inverse triplicate ratio of the distances from the central bodies; then the proportion of their absolute forces, measured in this manner, will vary as those equal distances vary. And if the law of the centripetal force in two different systems be as *any power of the distance*; (but a different power in the two systems) then there will always be one particular distance at which the centripetal force in each system will be equal; and therefore this can be no rule, whereby to determine the ratio of the absolute centripetal forces in any two such systems.

There are who look upon these things, respecting the logical part of the Newtonian Philosophy, as frivolous niceties. But is the Newtonian philosophy SCIENCE? Is it the regular deductions of REASON, from CLEAR PRINCIPLES? Is it what Newton himself says it is — "The whole science of motion accurately proposed and demonstrated: determining
" from

* Account of Newton's Discoveries, p. 108.

† In his Translation and Commentary on Newton.

“from any forces given, the motions that will result
 “from them; and conversely, from any motions
 “given, tracing the forces which are required to
 “produce them.”* Or, is the Newtonian philosophy like the philosophy of electricity, magnetism, &c. mere guesses work from a very slender analogy†. Men may amuse themselves with making curious experiments on such subjects. In time, when a very great number can be compared together, and with the other phænomena of nature; that *universal principle*, both of the coherence and motion of all bodies (which Newton ever supposed‡) may possibly be discovered. But a philosophy, term it Newtonian or what you will, which is no science, is no improvement of the understanding; nor worthy to be made the solemn object of pursuit, by the students in our UNIVERSITIES.

ON COR. I. PROP. X. AND COR. I. PROP. XIII. B. I.
 OF THE PRINCIPIA.

In Princip. Lib. I. Prop. X. Cor. I. Newton lays down the converse of Prop. X. and in Prop. XIII. Cor. I. he lays down the converse of Prop. XI. XII. XIII. The latter part of this last corollary, beginning, *Nam datis umbilico*, &c. is added in the last edition to obviate the objection of some learned foreigners, “that converse propositions required a proof.” The concise hint of a proof, here given by Newton, is much enlarged upon by the commentators *Le Seur* and *Jacquier*, who have been copied by others; but the reader is overwhelmed with algebra and geometry. Whenever the logical part of an argument

* Thorpe’s Translation of Newton’s Preface.

† That all we know on these subjects (at present) is no better, is evident from the many Theories of Electricity which have appeared one after another, and are forgotten. See a Theory of Electricity. Phil. Transf. 1748. And a Theory of Magnetism, by Dr. Knight.

‡ Preface to the Principia, near the end.

gument can be represented in plain words, without algebraic symbols, or geometrical diagrams; it is always clearer. And this we propose to do in what follows.

All the circumstances relating to a projectile, are *First*, the point from whence the body is projected; that is, its distance from the center of the forces; *Secondly*, the direction; *Thirdly*, the velocity with which the body is projected; and, *Fourthly*, the centripetal force at the point or place of projection. On these depends the nature of the orbit at the point of projection; which may differ from this orbit at another point, or from another orbit at a given point, in respect of the following circumstances. *First*, in the distance of the given point of the orbit from the center of the forces; *Secondly*, in the direction of the tangent (to the orbit) drawn through that point; that is, the angle which that tangent makes with a line drawn from the center of the forces to that given point; *Thirdly*, in the curvature of the orbit at the given point. The first of these depends upon the place from whence the body is projected. The second on the direction with which it is projected. The third depends on the centripetal force at the point of projection, in conjunction both with the velocity and direction of the projectile.

When all these are the same, the nature of the orbit at the given point must be the same; but the nature of the orbit at all subsequent points, will depend on the variation, or law of the centripetal force.

Now the FOCUS being given, *some one* of the three conic sections may be described, which shall pass through a point at any given distance from that focus; *Secondly*, shall touch a line (in that point) given in position; and, *Thirdly*, shall have a given curvature there. Therefore a conic section (*of some sort*) may be described, whose focus is the center of the forces, and which shall, at a given point, agree in all the circumstances of any orbit that can possibly be described,

scribed, in consequence of any supposed conditions under which a body can be projected. Hence, whatever those conditions are, the body at the point of projection may be, considered as moving in a part of one of the conic sections, whose focus is in the center of the forces.

But if the body can ever be supposed to be moving in any part of such a conic section; it will thence forward continue to move in that conic section; provided the centripetal force in other parts of the orbit, is to that force at the point of projection, in a reciprocal duplicate ratio of their distances from the center of attraction. For then the centripetal force is varied exactly in such a manner as is necessary to keep the body in that conic section, as appears from prop. XI. XII. XIII. For the purport of those propositions is, to find in what proportion the centripetal force must be varied, on a supposition that the body is *kept* thereby in the proposed orbit.

We said, *Some one* of the conic sections may be described, having the above properties — not *any one* of the three — which of the three depends on the given circumstances.

To propose a simple case: Let the tangent be perpendicular to a line drawn from the center of forces to the point of projection; then will the point of projection be the vertex of the conic section; because a line drawn from the focus is perpendicular to the tangent, at no other point but the vertex. Let now the radius of curvature be double the distance of the point of projection from the center of the forces; then must the orbit be a parabola. For the radius of curvature at the vertex of any conic section, is the semi-parameter. This in the ellipse is always less, in the parabola equal, in the hyperbola more than double the distance of the vertex from the focus.

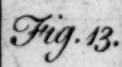
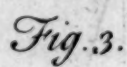
But further. An ellipse only may be described having a given CENTER, and which shall pass through a given point, touch a line (in that point) given in position,

sition, and have a given curvature there. Therefore under whatever circumstances a body may be projected, it may be supposed to begin to move in an ellipse, whose *center* is the center of the forces. Now if the force at the point of projection, varies in subsequent parts of the orbit in the direct ratio of their distance from the center; then the body will continue ever after in that same ellipse by Prop. X. And thus the converse of that proposition is demonstrated just as the converse of Prop. XI. XII. XIII. was demonstrated.

We will illustrate this by a case similar to what we had before. Let the tangent of projection, as before, be perpendicular to a line drawn from the center to the point of projection, then that point must be the extremity either of the first or second axis. For a line drawn from the center of an ellipse, is perpendicular to the tangent at no other point, but one of those two. Let, as before, the radius of curvature be double the distance of the point of projection from the center, then must that point be the extremity of the second axis. For, supposing the axes unequal; the semi-parameter, at the end of the first axis, is always less, at the end of the second axis, always greater, than the distance of those points from the center of the ellipse. When the two axes are equal (that is, when the ellipse becomes a circle) the semi-parameter is equal to that distance.

In the case here proposed, it is easy to see that the line drawn from the center of forces, to the point of projection, is the less semi-axis; and that the greater axis must be to the less as $\sqrt{2}$ to 1: for then the semi-parameter to the less axis, is double the less semi-axis, or double the distance of the point of projection, from the center of the forces, as was required.





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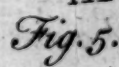


Fig. 2.



Fig. 3.

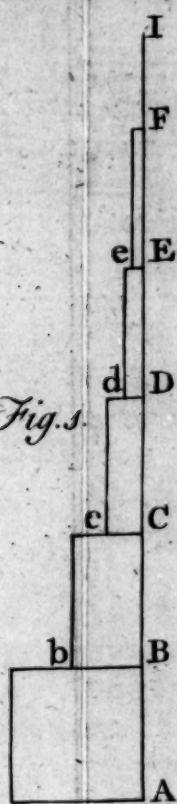
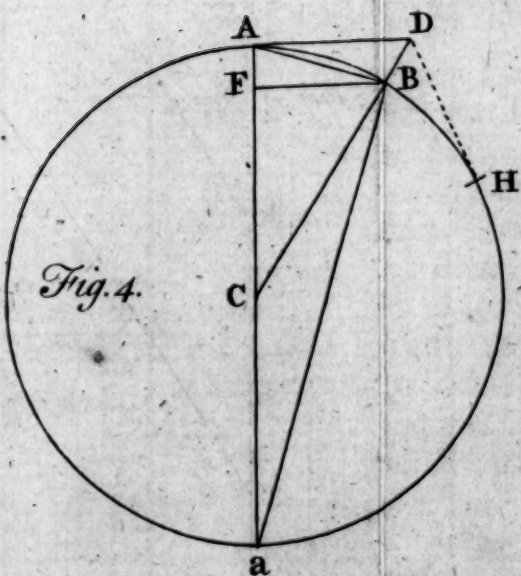


Fig. 4.



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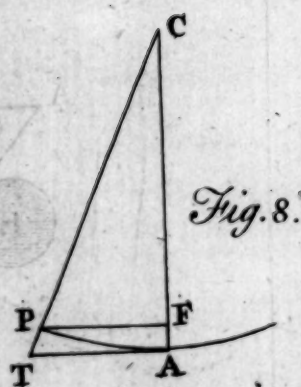
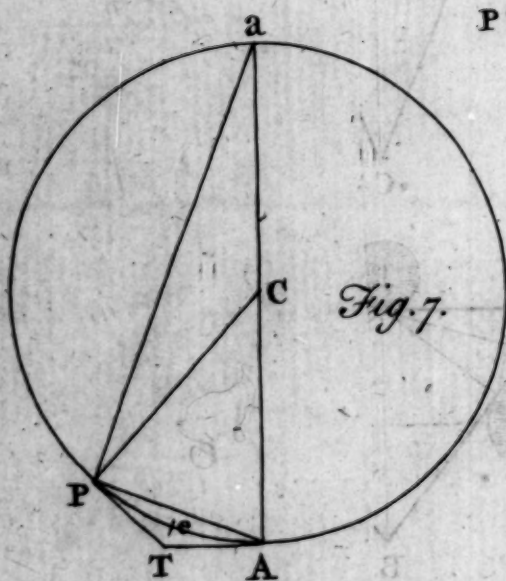
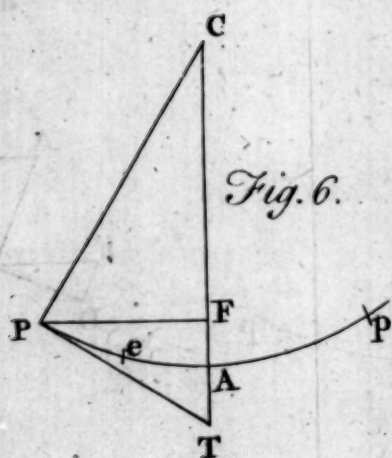
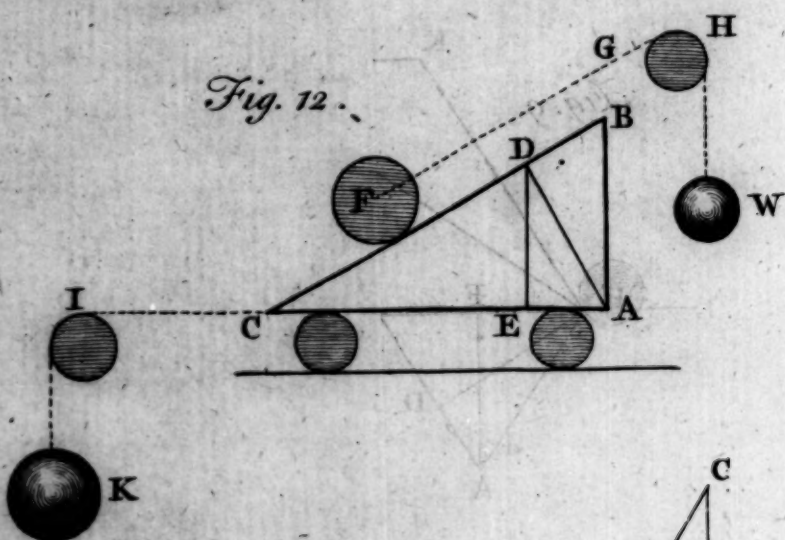


Fig. 9.

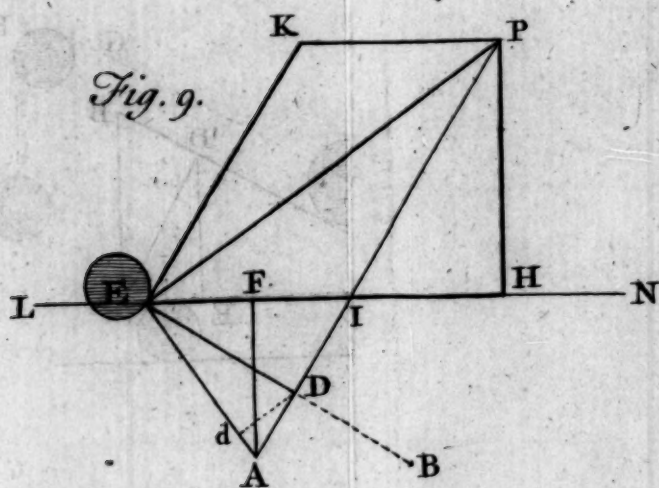


Fig. 10.

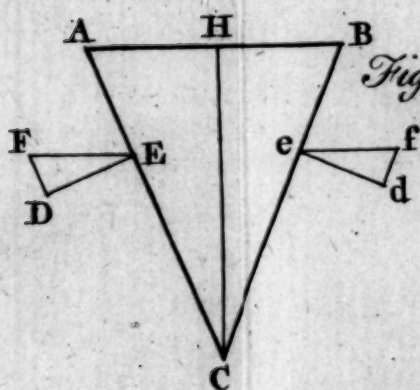
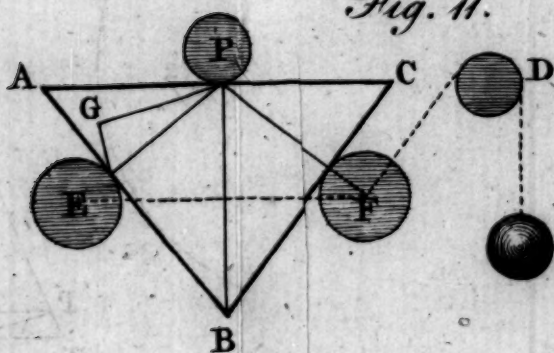


Fig. 11.



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